

HECKE-BOCHNER IDENTITY AND EIGENFUNCTIONS ASSOCIATED TO GELFAND PAIRS ON THE HEISENBERG GROUP

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ABSTRACT. Let \mathbb{H}^n be the $(2n+1)$ -dimensional Heisenberg group, and let K be a compact subgroup of $U(n)$, such that (K, \mathbb{H}^n) is a Gelfand pair. Also assume that the K -action on \mathbb{C}^n is polar. We prove a Hecke-Bochner identity associated to the Gelfand pair (K, \mathbb{H}^n) . For the special case $K = U(n)$, this was proved by Geller [6], giving a formula for the Weyl transform of a function f of the type $f = Pg$, where g is a radial function, and P a bigraded solid $U(n)$ -harmonic polynomial. Using our general Hecke-Bochner identity we also characterize (under some conditions) joint eigenfunctions of all differential operators on \mathbb{H}^n that are invariant under the action of K and the left action of \mathbb{H}^n .

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1. INTRODUCTION

This paper is concerned with two fundamental problems in Harmonic analysis on the Heisenberg group, \mathbb{H}^n . The first one is the Hecke-Bochner identity and the second one is a characterization of joint eigenfunctions for a certain family of invariant differential operators on \mathbb{H}^n . We first briefly recall the known results in this direction.

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The Hecke-Bochner identity on \mathbb{R}^n states that (see [15], Theorem-3.10, page-158) the Fourier transform of a function $f = Pg$, where P is a homogeneous solid $SO(n)$ -harmonic polynomial (of degree k say) and g is radial, is given by $\widehat{Pg} = Ph$, where h is a radial function given by

$$h(r) = i^{-k} \int_{s=0}^{\infty} g(s) \frac{J_{\frac{n}{2}+k-1}(rs)}{(rs)^{\frac{n}{2}+k-1}} s^{n+k-1} ds,$$

where $J_{\frac{n}{2}+k-1}$ is the Bessel's function of order $\frac{n}{2} + k - 1$. Secondly, any eigenfunction φ of Δ , the Laplacian on \mathbb{R}^n , with eigenvalue $-\lambda^2$ is given by the integral representation

$$\varphi(x) = \int_{S^{n-1}} e^{i\lambda x \cdot \omega} dT(\omega),$$

where T is a certain analytic functional. See Helgason ([8], Theorem 2.1, page-5) for $n = 2$ and Hashizume et al [7] for general case. Both these results can be interpreted in terms of harmonic analysis on the Gelfand pair $(\mathbb{R}^n \ltimes SO(n), SO(n))$. Note that a solid homogeneous harmonic polynomial of degree k is an element which transforms according to a class one representation of $SO(n)$. Next, the Laplacian Δ is the generator of $\mathbb{R}^n \ltimes SO(n)$ invariant differential operators on \mathbb{R}^n . This point of views have a natural generalization to other homogeneous spaces.

In the context of Riemannian symmetric spaces $X = G/K$, Helgason ([11], Corollary 7.4) characterized all K -finite joint eigenfunctions for $D(G/K)$. The characterization of arbitrary joint eigenfunctions for $D(G/K)$ was done by Helgason ([10], Chapter IV, Corollary 1.6) when $\text{rank} X = 1$ and by Kashiwara et al [12] in the general case. A Hecke-Bochner type identity was established, when X is of rank one, by Bray [3]. For general case see [9], Chapter-III, Corollary 5.5.

In this paper, we consider these two questions on the Heisenberg group associated to the Gelfand pair (K, \mathbb{H}^n) , where $K \subset U(n)$ and the K -action on \mathbb{C}^n is polar. We prove a Hecke-Bochner type identity (Theorem 7.4), giving a formulae for the Weyl transform of a function which transforms according to a class one representation of K . We will see that the formulae involves generalized K -spherical functions, as in the case of Euclidean spaces and Riemannian symmetric spaces. For the special

case $K = U(n)$ this was already proved by Geller ([6], Theorem 4.2). Let $\mathcal{L}_K(\mathbf{h}_n)$ be the algebra of all differential operators on \mathbb{H}^n that are invariant under the action of K and the left action of \mathbb{H}^n . Any joint eigenfunction of all $D \in \mathcal{L}_K(\mathbf{h}_n)$ has to be of the form $f(z, t) = e^{i\lambda t}g(z)$ for some complex number λ . Following the view point of Thangavelu in [16], under the assumptions that λ is non-zero real and $e^{-(|\lambda|-\epsilon)|z|^2}|g(z)| \in L^p(\mathbb{C}^n)$ for some $\epsilon > 0$ and $1 \leq p \leq \infty$, we characterize all K -finite joint eigenfunctions $f(z, t)$ of all $D \in \mathcal{L}_K(\mathbf{h}_n)$, in terms of the representations of the Heisenberg group (Theorem **8.3**). We extend this result for arbitrary (with the same growth condition) joint eigenfunctions, when $\dim V_\delta^M = 1$ for all class one representations δ of K ; here M is the stabilizer of a K -regular point, V_δ is the (finite dimensional) Hilbert space where the representation δ is realized and V_δ^M is the space of M -fixed vectors in V_δ . This can be put in a different form, giving an integral representation of eigenfunctions, which for $K = U(n)$ is precisely Theorem 4.1 in [16]. We also obtain a different integral representation with an explicit kernel.

The plan of the paper is as follows. In section 2., we recall the definition of polar action of $K \subset SO(n)$ on \mathbb{R}^n , develop a system of polar coordinates and state some results about polar actions. In section 3., we show that the Kostant-Rallis Theorem holds for polar actions i.e each K -harmonic polynomial is determined by its values on a regular K -orbit. We also discuss the class one representations of K realized on the space of K -harmonic polynomials and on the space of their restriction to a regular K -orbit. In section 4., for a class one representation δ of K , we consider δ type $\text{Hom}(V_\delta, V_\delta)$ -valued functions G i.e $G : \mathbb{R}^n \rightarrow \text{Hom}(V_\delta, V_\delta)$ such that $G(k \cdot x) = \delta(k)G(x)$. We show that such a G can be written in a special form, which, for the case $K = SO(n)$, is equivalent to considering a function of type Pg , where P is a solid homogeneous $SO(n)$ -harmonic polynomial of certain degree and g is radial. In section 5., we mainly recall some basic facts related to the Heisenberg group, its representations and Weyl transform. We also state some results about Gelfand pairs and bounded K -spherical functions from [2]. Section 6., deals with the Weyl transform of K -invariant functions. In section 7., we prove the

main results of this paper. We start with defining generalized K -spherical functions, prove a Hecke-Bochner type identity for the Weyl transform. Using this we prove the uniqueness (upto a right multiplication by a constant matrix) of generalized K -spherical functions. We also give a formulae of generalized K -spherical function in terms of the representations of Heisenberg group. This formulae together with the uniqueness of generalized K -spherical functions will imply characterizations of K -finite joint eigenfunctions (with the usual growth condition) of all $D \in \mathcal{L}_K(\mathfrak{h}_n)$, which we present in section 8. Section 9. deals with square integrable (modulo the center) joint eigenfunctions. In the final section, we discuss the special case when $\dim V_\delta^M = 1$ for all class one representations δ of K .

2. POLAR ACTIONS AND COORDINATES

In this section we recall polar actions and develop a system of polar coordinates on the spaces upon they act. References for this section are Conlon [4], Dadok [5] and Lander [13]. Let K be a compact connected subgroup of $SO(n)$ which acts naturally on \mathbb{R}^n . Let \mathfrak{k} be the Lie algebra of K . We denote the inner product on \mathbb{R}^n by (\cdot, \cdot) . Let $N_x := \{k \cdot x : k \in K\}$ be the K -orbit through x , and $K_x := \{k : k \cdot x = x\}$ be the isotropy subgroup of x , hence $N_x \cong K/K_x$. A K -orbit of maximal dimension is called a *regular orbit*, and any point on a regular orbit is called a *regular point*. A K -orbit through a point x is called a *principal orbit* if K_x is a subgroup of a conjugate of any other isotropy subgroup. Clearly any principal orbit is also a regular orbit. The action of K on \mathbb{R}^n is called *polar action* if there is a linear subspace T of \mathbb{R}^n which meets every K -orbit and is orthogonal to the K -orbit at every point i.e $(\mathfrak{k} \cdot x, T) = 0$ for all $x \in T$. Such a linear subspace T is called a K -*transversal domain*. This is precisely the condition (A) in the introduction of [4]. Then $\dim(T) = \dim(\mathbb{R}^n) - \dim$ of a regular orbit ([4], Proposition 1.1). Therefore if we take a regular point $x \in T$ then clearly $A_x = T$ where $A_x = \{y \in \mathbb{R}^n : (y, \mathfrak{k} \cdot x) = 0\}$. Consequently A_x meets all the orbits orthogonally. Hence the above definition of polar action is equivalent to that of Dadok [5]. Also, for polar action any orbit of maximal

dimension is principal ([4], Proposition 2.2). Therefore, regular orbits and principal orbits are equivalent for polar action.

From now on we always assume that K is a compact connected subgroup of $SO(n)$ whose action on \mathbb{R}^n is polar. We state some results from Conlon [4] and derive some easy consequences. Since regular orbits and principal orbits are same, we only use the word “regular orbit” instead of using both. As mentioned above we have,

Proposition 2.1. (Conlon [4], Proposition 1.1) *Let $N \subset \mathbb{R}^n$ be a K -orbit of maximal dimension. Then $\dim(N) = \dim(\mathbb{R}^n) - \dim(T)$.*

Theorem 2.2. (Conlon [4], Theorem II) *Let $T \subset \mathbb{R}^n$ be a K -transversal domain. Then there is a finite collection P_1, P_2, \dots, P_r of hyperplanes in T , together with positive integers $m(i)$, $i = 1, 2, \dots, r$, such that for each $x \in T$,*

$$\dim(N_x) = \dim(\mathbb{R}^n) - \dim(T) - \sum_{i \in I_x} m(i),$$

where $I_x = \{i : x \in P_i\}$.

Definition 2.3. Each P_i as above is called a *singular variety* of multiplicity $m(i)$, and each connected component of $T \setminus \cup P_i$ a *Weyl domain* in T . The *Weyl group* $W = W(K, T)$ is the group of transformations of T consisting of those $k \in K$ such that $k \cdot T = T$.

Theorem 2.4. (Conlon [4], Theorem III) *If T is a K -transversal domain, then the orthogonal reflection of T in each singular variety P_i exists, W is a finite group generated by all such reflections, and W permutes simply transitively the set of Weyl domains in T . If $x \in T$ lies on no singular variety, then W permutes simply transitively the set $N_x \cap T$.*

Fix a Weyl domain T^+ in T . As an easy consequence of the above three results we get the following corollary.

Corollary 2.5. *All the points of T^+ are regular, and each regular K -orbit intersects T^+ exactly at one point.*

Lemma 2.6. *If $x \in T$ is regular then $K_x = K_T$, where $K_T := \{k \in K : k \cdot q = q, \forall q \in T\}$ is the stabilizer of T .*

Proof. See the proof of Proposition 1.1 in Conlon [4]. \square

Let $M = K_T$ be as defined in the above lemma. Define the “polar coordinate mapping”

$$\phi : T^+ \times K/M \longrightarrow \mathbb{R}^n \text{ by } \phi(r, kM) = k \cdot r.$$

Clearly ϕ is well defined and by Corollary 2.5, its image is precisely the set of all regular points. If $k_1 \cdot r_1 = k_2 \cdot r_2$ for $k_1, k_2 \in K$ and $r_1, r_2 \in T^+$, then K -orbits through r_1 and r_2 are same. By Corollary 2.5, $r_1 = r_2 = r$ (say). Consequently $k_1^{-1}k_2$ fixes r and hence belongs to K_T by Lemma 2.6. Therefore $(r_1, k_1M) = (r_2, k_2M)$. So, we have proved the following proposition.

Proposition 2.7. *The polar coordinate mapping ϕ , defined above, is a bijection of $T^+ \times K/M$ onto the set of regular points in \mathbb{R}^n whose complement has measure zero.*

For a regular point x , if $x = \phi(r, kM)$ for $r \in T^+$ and $k \in K$ then we simply write $x = (r, kM)$ and call this the polar coordinates of x . It is clear from the definition of ϕ , that $k_1 \cdot (r, k_2M) = (r, k_1k_2M)$, $r \in T^+$ and $k_1, k_2 \in K$.

Remark 2.8. Let $K = SO(n)$. Consider the K -regular point $e_1 = (1, 0, 0, \dots, 0) \in \mathbb{R}^n$. A K -transversal domain T can be chosen to be $T = A_{e_1} = \{(x, 0, 0, \dots, 0) \in \mathbb{R}^n : x \in \mathbb{R}\}$, and $T^+ = \{(r, 0, 0, \dots, 0) \in \mathbb{R}^n : r > 0\}$, which can be identified with $(0, \infty)$. If M is the stabilizer of e_1 , via the map $kM \rightarrow k \cdot e_1$, we have the identification $K/M = K \cdot e_1 = S^{n-1}$. Therefore, by the above proposition, it follows that each regular point $x \in \mathbb{R}^n$ can be written uniquely as $x = (r, \omega) = r\omega$, $r > 0, \omega \in S^{n-1}$, which gives the usual polar coordinate system on \mathbb{R}^n .

We conclude this section by relating polar actions and symmetric space actions, due to Dadok [5]. First we define a symmetric space action.

Definition 2.9. The action of a connected subgroup G of $SO(n)$ with Lie algebra \mathfrak{g} on \mathbb{R}^n is called a symmetric space action if there is a real semisimple Lie algebra \mathfrak{u} with Cartan decomposition $\mathfrak{u} = \mathfrak{k}' + \mathfrak{p}$, a Lie algebra isomorphism $A : \mathfrak{g} \longrightarrow \mathfrak{k}'$, and a real vector space isomorphism $L : \mathbb{R}^n \longrightarrow \mathfrak{p}$ such that $L(X \cdot y) = [A(X), L(y)]$ for all $X \in \mathfrak{g}$ and $y \in \mathbb{R}^n$. Here $[\cdot, \cdot]$ denotes the Lie algebra bracket on \mathfrak{g} .

Remark 2.10. Let the action of G be a symmetric space action. If U is a connected Lie group with Lie algebra \mathfrak{u} , and K' is a connected subgroup of U with Lie algebra \mathfrak{k}' , then the action of G on \mathbb{R}^n is isomorphic to that of $\text{Ad}(K')$ on \mathfrak{p} , i.e if we identify \mathbb{R}^n and \mathfrak{p} via the map L , then G -orbits and $\text{Ad}(K')$ -orbits coincide.

The relation between a polar and a symmetric space action is provided by the following proposition.

Proposition 2.11. (Dadok [5], Proposition 6) *Let K be a connected, compact subgroup of $SO(n)$ whose action on \mathbb{R}^n is polar. Then there exists a connected subgroup G of $SO(n)$ whose action on \mathbb{R}^n is a symmetric space action and whose orbits coincide with those of K .*

3. K -HARMONIC POLYNOMIALS

Throughout this section we assume that K is a connected compact subgroup of $SO(n)$ whose action on \mathbb{R}^n is polar, T a K -transversal domain, and $M = K_T$, the centralizer of T . Let S denote the space of polynomials on \mathbb{R}^n , $I \subset S$ the set of K -invariants in S and I_+ the set of polynomials in I without the constant term. Let $H \subset S$ denote the set of K -harmonic polynomials, that is, polynomials annihilated by the constant coefficient differential operators on \mathbb{R}^n defined by elements in I_+ . For more details about K -harmonic polynomials see Helgason [8], Chapter III. The following result is proved there (Theorem 1.1).

Theorem 3.1. *$S=IH$, that is, each polynomial p on \mathbb{R}^n has the form $p = \sum_k i_k h_k$ where i_k is K -invariant and h_k is K -harmonic.*

Since the action of K is polar by Proposition 2.11, there is a connected subgroup G of $SO(n)$ whose action on \mathbb{R}^n is a symmetric space action and whose orbits coincide with those of K . Let L , K' and \mathfrak{p} are as in Definition 2.9. Therefore, by Remark 2.10, if we identify \mathbb{R}^n and \mathfrak{p} via the map L , then K and $\text{Ad}(K')$ orbits coincide. Hence I and I_+ are same for both actions and consequently so is H . So, for polar actions we have the following version of Kostant-Rallis Theorem (see Helgason [9], Chapter III, Theorem 2.4).

Theorem 3.2. *Each K -harmonic polynomial is determined by its values on a regular K -orbit.*

Now we briefly describe the class one representations of K realized on the space H and on the space of their restriction to a regular K -orbit. This is similar to the symmetric space theory (see Helgason [9], page-236,237 and 298,299; [8], page-533). For $x \in T$ regular, consider the embedding $K/M = N_x \subset \mathbb{R}^n$ via the map $kM \longrightarrow k \cdot x$. Then as is well known (Helgason [8], Exercise A1 (iv), page-73) each K -finite function on K/M is the restriction of a polynomial $p \in S$ which by Theorem 3.1 can be taken to be harmonic. Thus, by Theorem 3.2 we see that the restriction mapping $h \longrightarrow h|_{N_x}$ is a bijection of H onto the space of K -finite functions in $\mathcal{E}(K/M)$ (the space of smooth functions on K/M). Let \widehat{K}_M be the set of all inequivalent unitary irreducible representation of K having M fixed vector. If $\delta \in \widehat{K}_M$, let H_δ (respectively $\mathcal{E}_\delta(K/M)$) denote the space of K -finite functions in H (respectively $\mathcal{E}(K/M)$) of type δ . Then the restriction mapping maps H_δ onto $\mathcal{E}_\delta(K/M)$. Let V_δ be the (finite dimensional) Hilbert space on which δ is realized and let $V_\delta^M \subset V_\delta$ be the space of M -fixed vectors. Let $v_1, v_2, \dots, v_{d(\delta)}$ be an orthonormal basis of V_δ such that $v_1, v_2, \dots, v_{l(\delta)}$ span V_δ^M . Then the functions

$$kM \longrightarrow \langle v_j, \delta(k)v_i \rangle \quad 1 \leq j \leq d(\delta), \quad 1 \leq i \leq l(\delta)$$

form a basis of $\mathcal{E}_\delta(K/M)$ (Theorem 3.5, chapter V, Helgason [8]), and

$$\mathcal{E}_\delta(K/M) = \bigoplus_{i=1}^{l(\delta)} \mathcal{E}_{\delta,i}(K/M), \quad (3.1)$$

where $\mathcal{E}_{\delta,i}(K/M)$ is the space of functions

$$F_{v,i}(K/M) = \langle v, \delta(k)v_i \rangle, \quad v \in V_\delta.$$

The map $v \longrightarrow F_{v,i}$ is an isomorphism of V_δ onto $\mathcal{E}_{\delta,i}(K/M)$ commuting with the action of K . Consequently H_δ decomposes into $l(\delta)$ copies of δ . Thus we write

$$H_\delta = \bigoplus_{i=1}^{l(\delta)} H_{\delta,i}, \quad (3.2)$$

where the action of K on each $H_{\delta,i}$ is equivalent to δ (by decomposing H_δ first into homogeneous components we can assume that the $H_{\delta,i}$ consists of homogeneous polynomials of degree say $d_i(\delta)$), and the vector space $F_\delta = \text{Hom}_K(V_\delta, H_\delta)$ of linear maps η of V_δ into H_δ satisfying

$$\eta(\delta(k)v) = k \cdot (\eta(v)) \quad k \in K, \quad v \in V_\delta \quad (3.3)$$

has dimension $l(\delta)$.

Remark 3.3. Let $K = SO(n)$. Let e_1, T, T^+, M be as in the Remark 2.8. Also we have the identification $K/M = S^{n-1}$. In this special case, note that, the space H consists of all polynomials P such that $\Delta P = 0$, where $\Delta = \Sigma \partial^2 / \partial x_i^2$ is the usual Laplacian on \mathbb{R}^n . Let \mathcal{H}_m denotes the space of all m th degree homogeneous polynomials in H and \mathcal{S}_m denotes the space of restrictions of elements of \mathcal{H}_m to S^{n-1} . The elements of \mathcal{H}_m are called solid harmonics of degree m , and those of \mathcal{S}_m are called spherical harmonics of degree m . The K -action on $K/M = S^{n-1}$ defines a unitary representation on $L^2(S^{2n-1})$. Clearly each \mathcal{S}_m is a K -invariant subspace. Let δ_m denotes the restriction of δ to \mathcal{S}_m . In fact these describe all inequivalent, irreducible, unitary representations in \widehat{K}_M . Note that according to our general notation, $H_{\delta_m} = \mathcal{H}_m$, $\mathcal{E}_{\delta_m}(K/M) = \mathcal{S}_m$, and $l(\delta_m) = \dim V_{\delta_m}^M = 1$. Let v^m be the unique (upto constant multiple) unit M -fixed vector in V_{δ_m} . Then the one dimensional vector space $F_{\delta_m} = \text{Hom}(V_{\delta_m}, \mathcal{H}_m)$ is spanned by the linear map $\eta_{\delta_m} : V_{\delta_m} \rightarrow \mathcal{H}_m$, where for $v \in V_{\delta_m}$, $\eta_{\delta_m}(v)$ is the unique element in \mathcal{H}_m whose restriction to S^{n-1} is $Y_v(kM) := \langle v, \delta_m(k)v^m \rangle \in \mathcal{S}_m$ i.e $\eta_{\delta_m}(v)(x) = |x|^m Y_v(x/|x|)$.

4. K-TYPE FUNCTIONS IN MATRIX FORM

We assume that K is a connected, compact subgroup of $SO(n)$ whose action on \mathbb{R}^n is polar. We use all the notation from the previous two sections. For two finite dimensional vector spaces V and W denote the space of all linear maps from V into W , by $\text{Hom}(V, W)$. For two positive integers p and q denote the space of all $p \times q$ matrices with complex entries by $\mathcal{M}_{p \times q}$. If A is a set and $f : A \longrightarrow \mathcal{M}_{p \times q}$ a function, then we define $f_{ij} : A \longrightarrow \mathbb{R}^n$ by $f_{ij}(a) = (i, j)\text{th entry of } f(a)$, for $a \in A$. For $\delta \in \widehat{K}_M$ define $\mathcal{X}^\delta(\mathbb{R}^n)$ to be the set of all functions

$$F : \mathbb{R}^n \longrightarrow \text{Hom}(V_\delta^M, V_\delta)$$

satisfying the condition

$$F(k \cdot x) = \delta(k)F(x) \quad \forall x \in \mathbb{R}^n, k \in K, \quad (4.1)$$

and $\mathcal{Y}^\delta(\mathbb{R}^n)$ to be the set of all functions

$$G : \mathbb{R}^n \longrightarrow \text{Hom}(V_\delta, V_\delta)$$

satisfying the conditions

$$G(k \cdot x) = \delta(k)G(x), \quad G(x)\delta(m) = G(x) \quad \forall x \in \mathbb{R}^n, k \in K, m \in M. \quad (4.2)$$

Here the multiplications are the compositions of linear maps. Proposition **3.1** below says that the sets $\mathcal{X}^\delta(\mathbb{R}^n)$ and $\mathcal{Y}^\delta(\mathbb{R}^n)$ can be identified. Also, define $\mathcal{E}^\delta(\mathbb{R}^n)$ to be the space of all smooth functions in $\mathcal{X}^\delta(\mathbb{R}^n)$. Choose an orthonormal ordered basis $\mathbf{b} = \{v_1, v_2, \dots, v_{d(\delta)}\}$ for V_δ , so that $\mathbf{b}^M = \{v_1, v_2, \dots, v_{l(\delta)}\}$ form an ordered basis for V_δ^M . Identify δ with its matrix representation with respect to the basis \mathbf{b} . Then we can identify $\mathcal{X}^\delta(\mathbb{R}^n)$ with the space of all functions

$$F : \mathbb{R}^n \longrightarrow \mathcal{M}_{d(\delta) \times l(\delta)}$$

satisfying (4.1) (but now, the multiplications are simply matrix multiplications), via the matrix representation with respect to bases \mathbf{b} for V_δ and \mathbf{b}^M for V_δ^M . Similarly identify $\mathcal{Y}^\delta(\mathbb{R}^n)$ and $\mathcal{E}^\delta(\mathbb{R}^n)$ with their corresponding matrix representations with

respect to bases \mathbf{b} and \mathbf{b}^M . Through out this paper, we use these identifications with respect to the basis \mathbf{b} and \mathbf{b}^M . Define

$$Y^\delta : K/M \longrightarrow \mathcal{M}_{d(\delta) \times l(\delta)}$$

by

$$Y_{ij}^\delta(kM) = \delta_{ij}(k) = \langle \delta(k)v_j, v_i \rangle, \quad 1 \leq i \leq d(\delta), \quad 1 \leq j \leq l(\delta).$$

If $\check{\delta}$ denote the contragredient representation, choose $V_{\check{\delta}} = V_\delta^*$ (the dual vector space of V_δ) with inner product \langle, \rangle defined by $\langle v^*, w^* \rangle = \langle w, v \rangle$, $v, w \in V_\delta$. Take the orthonormal ordered basis \mathbf{b}^* of $V_{\check{\delta}}$ to be the dual basis $\{v_1^*, v_2^*, \dots, v_{d(\delta)}^*\}$. Then $\mathbf{b}^{*M} = \{v_1^*, v_2^*, \dots, v_{l(\delta)}^*\}$ will be a basis for $V_{\check{\delta}}^M$. Identify $\check{\delta}$ with its matrix representation with respect to the basis \mathbf{b}^* . Then $\check{\delta}_{ij}(k) = \overline{\delta_{ij}(k)}$. Therefore $\{Y_{ij}^\delta(kM) : 1 \leq i \leq d(\delta), 1 \leq j \leq l(\delta)\}$ form a basis for $\mathcal{E}_{\check{\delta}}(K/M)$. For more details about contragredient representation see Helgason [8], page-393,533. Now, take an ordered basis $\mathbf{e} = \{\eta_1, \eta_2, \dots, \eta_{l(\delta)}\}$ for $F_{\check{\delta}} = \text{Hom}_K(V_{\check{\delta}}, H_{\check{\delta}})$. Define

$$P^\delta : \mathbb{R}^n \longrightarrow \mathcal{M}_{d(\delta) \times l(\delta)}$$

by

$$P_{ij}^\delta(x) = \eta_j(v_i^*)(x), \quad 1 \leq i \leq d(\delta), \quad 1 \leq j \leq l(\delta).$$

Since $\eta_j(\check{\delta}(k)v_i^*) = k \cdot (\eta_j(v_i^*))$, using the fact that $\check{\delta}_{ij}(k) = \overline{\delta_{ij}(k)}$ and $\check{\delta}$ is unitary, one can show that $P^\delta(k \cdot x) = \delta(k)P^\delta(x)$. Hence $P^\delta \in \mathcal{E}^\delta(\mathbb{R}^n)$. Define

$$\Upsilon_\delta : \mathbb{R}^n \longrightarrow \mathcal{M}_{l(\delta) \times l(\delta)}$$

by

$$\Upsilon_\delta(x) = [P^\delta(x)]^* [P^\delta(x)]. \quad (4.3)$$

Here \star denotes the matrix adjoint. Clearly Υ_δ is K -invariant.

Remark 4.1. Let $K = SO(n)$. We describe Y^δ , P^δ and Υ_δ in this special case. Let e_1, T, T^+, M be as in the Remark 2.8, and $\mathcal{H}_m, \mathcal{S}_m, \delta_m, v^m, \eta_{\delta_m}$ be as in Remark 3.3. Choose an ordered orthonormal basis $\{v_1, v_2, \dots, v_{d(m)}\}$ for V_{δ_m} , such that $\{v_1 = v^m\}$

is the orthonormal basis for $V_{\delta_m}^M$. Then $\{Y_{i1}^{\delta_m}(kM) = \langle \delta_m(k)v_1, v_i \rangle : 1 \leq i \leq d(m)\}$ forms an orthogonal basis for $\mathcal{E}_{\delta_m}^\vee(K/M) = \bar{\mathcal{S}}_m = \mathcal{S}_m$, and $\sum_{i=1}^{d(\delta)} |Y_{i1}^{\delta_m}(kM)|^2 = 1$. Take $\{\eta_{\delta_m}^\vee\}$ as a basis for $F_{\delta_m}^\vee$. Then, by Remark 3.3, for $x = (r, kM) = (r, \omega)$,

$$P_{i1}^{\delta_m}(x) = \eta_{\delta_m}^\vee(v_i^*)(x) = r^m \langle v_i^*, \delta_m^\vee(k)v_1^* \rangle = r^m \langle \delta_m(k)v_1, v_i \rangle = r^m Y_{i1}^{\delta_m}(kM) = |x|^m Y_{i1}^{\delta_m}(\omega)$$

i.e $P_{i1}^{\delta_m}$ is the unique element in \mathcal{H}_m whose restriction to S^{n-1} is $Y_{ij}^{\delta_m}$.

From the above discussion we can prove the following : Take $P_m^i \in \mathcal{H}_m$, and $Y_m^i \in \mathcal{S}_m$ to be their restrictions to S^{n-1} so that $\{Y_m^i : i = 1, 2, \dots, d(m)\}$ forms an orthonormal basis for \mathcal{S}_m . Then it is possible to choose orthonormal ordered bases $\mathbf{b} = \{v_1, v_2, \dots, v_{d(m)}\}$ for V_{δ_m} and $\mathbf{b}^M = \{v_1\}$ for $V_{\delta_m}^M$, so that, with respect to these bases, $Y^{\delta_m} : S^{n-1} \rightarrow \mathcal{M}_{d(m) \times 1}$ is given by

$$Y^{\delta_m}(\omega) = \sqrt{\frac{|S^{n-1}|}{d(m)}} \left[Y_m^1(\omega), Y_m^2(\omega), \dots, Y_m^{d(m)}(\omega) \right]^t, \omega \in S^{n-1}. \quad (4.4)$$

We can choose a basis \mathbf{e} for $F_{\delta_m}^\vee$ so that, $P^{\delta_m} : \mathbb{R}^n \rightarrow \mathcal{M}_{d(m) \times 1}$ is given by

$$P^{\delta_m}(x) = \sqrt{\frac{|S^{n-1}|}{d(m)}} \left[P_m^1(x), P_m^2(x), \dots, P_m^{d(m)}(x) \right]^t, x \in \mathbb{R}^n. \quad (4.5)$$

In particular,

$$P^{\delta_m}(x) = |x|^m Y^{\delta_m}(x/|x|). \quad (4.6)$$

Also, we have

$$\Upsilon_{\delta_m}(x) = \frac{|S^{n-1}|}{d(m)} \sum_{i=1}^{d(m)} \overline{P_m^i(x)} P_m^i(x) = \frac{|S^{n-1}|}{d(m)} |x|^{2m} \sum_{i=1}^{d(m)} |Y_m^i(\omega)|^2 = |x|^{2m}.$$

Proposition 4.2. *Each $G \in \mathcal{Y}^\delta(\mathbb{R}^n)$ is determined by its restriction on V_δ^M .*

Proof. If $G \in \mathcal{Y}^\delta(\mathbb{R}^n)$, then G is identified with its $(d(\delta) \times d(\delta))$ matrix with respect to the fixed basis \mathbf{b} . Hence it is enough to show that all the entries in last $(d(\delta) - l(\delta))$ columns of G are zero. Since $G(x)\delta(m) = G(x)$ for all $m \in M$, equating the matrix entries on both sides we get, for $1 \leq i, j \leq d(\delta)$,

$$G_{ij}(x) = \sum_{p=1}^{d(\delta)} G_{ip}(x) \delta_{pj}(m), \quad \forall m \in M. \quad (4.7)$$

Since $v_j \in (V_\delta^M)^\perp$ for $j \geq l(\delta)$, $\int_M \delta(m) v_j dm = 0$ if $j \geq l(\delta)$. So,

$$\int_M \delta_{pj}(m) dm = \int_M \langle \delta(m) v_j, v_p \rangle dm = 0, \quad 1 \leq p \leq d(\delta), \quad j \geq l(\delta). \quad (4.8)$$

Therefore for $j \geq l(\delta)$, integrating both side of (4.7) over M we get the desired result. \square

Lemma 4.3. *Suppose F is in $\mathcal{E}^\delta(\mathbb{R}^n)$. Then there is a unique function $G_0 : T^+ \rightarrow \mathcal{M}_{l(\delta) \times l(\delta)}$ such that for all regular points $x = (r, kM)$,*

$$F(x) = Y^\delta(kM) G_0(r).$$

Proof. First note that the uniqueness follows from the fact that $Y^\delta(kM)$ has a left inverse namely $[Y^\delta(kM)]^*$. Since $F(\sigma \cdot x) = \delta(\sigma) F(x)$ for all $\sigma \in K$, we can write (for $x = (r, kM)$ regular)

$$\begin{aligned} F(x) &= \int_K \delta(\sigma)^{-1} F(\sigma \cdot x) d\sigma \\ &= \int_K \delta(\sigma)^{-1} F(\sigma \cdot (r, kM)) d\sigma \\ &= \int_K \delta(\sigma)^{-1} F(r, \sigma kM) d\sigma \\ &= \int_K \delta(\sigma k^{-1})^{-1} F(r, \sigma M) d\sigma \\ &= \delta(k) \int_K \delta(\sigma)^{-1} F(r, \sigma M) d\sigma \\ &= \delta(k) \int_K \int_M \delta(\sigma m)^{-1} F(r, \sigma M) d\sigma dm = \delta(k) G'_0(r), \end{aligned}$$

where

$$G'_0(r) = \int_K \int_M \delta(\sigma m)^{-1} F(r, \sigma M) d\sigma dm.$$

Now,

$$\delta_{ij}(\sigma m) = \sum_{p=1}^{d(\delta)} \delta_{ip}(\sigma) \delta_{pj}(m).$$

Integrating both sides over M and using (4.8) we get (for each $\sigma \in K$),

$$\int_M \delta_{ij}(\sigma m) dm = 0, \quad 1 \leq i \leq d(\delta), \quad j \geq l(\delta).$$

Since $\delta(\sigma m)^{-1} = \overline{\delta(\sigma m)}^t$, all the entries in last $(d(\delta) - l(\delta))$ rows of the matrix $\int_M \delta(\sigma m)^{-1} dm$ are zero for all $\sigma \in K$, and consequently so is for the $d(\delta) \times l(\delta)$ matrix

$$G'_0(r) = \int_K \int_M \delta(\sigma m)^{-1} F(r, \sigma M) d\sigma dm$$

(note that F is a $(d(\delta) \times l(\delta))$ matrix). Therefore, if we define $G_0 : T^+ \longrightarrow \mathcal{M}_{l(\delta) \times l(\delta)}$ by

$$(G_0)_{ij}(r) = (G'_0)_{ij}(r), \quad 1 \leq i, j \leq l(\delta),$$

then

$$\delta(k)G'_0(r) = Y^\delta(kM)G_0(r),$$

since first $l(\delta)$ columns in the matrix $\delta(k)$ are precisely the columns in $Y^\delta(kM)$. Hence the proof. \square

Corollary 4.4. *Let $F \in \mathcal{E}^\delta(\mathbb{R}^n)$. Then the j th column of F is determined by F_{1j} . In particular F is determined by its first row.*

Proof. Let $F \in \mathcal{E}^\delta(\mathbb{R}^n)$ be such that all the entries in first row are identically zero. We have to show that $F \equiv 0$. Let G_0 be as in the previous lemma. We have

$$\sum_{p=1}^{l(\delta)} Y_{1p}^\delta(kM)(G_0)_{pj}(r) = 0 \text{ for all regular points } (r, kM).$$

Since Y_{1p}^δ s are linearly independent, for each $r \in T^+$, $(G_0)_{pj}(r) = 0$, $p = 1, 2, \dots, l(\delta)$; and consequently the j th column of F is zero on the set of regular points. The set of regular points being dense in \mathbb{R}^n , we are done. \square

Remark 4.5. Using the above arguments, one can show the following : For $F \in \mathcal{E}^\delta(\mathbb{R}^n)$, let V_F denote the finite dimensional vector space spanned by F_{ij} 's, and V_F^i denote the space spanned by the entries of i th row in F . Let $m(\delta)$ be the number of linearly independent columns in F . Also assume that first $m(\delta)$ columns are linearly independent. Then $\{F_{ij} : j = 1, 2, \dots, m(\delta)\}$ form a basis for V_F^i ; and $\{F_{ij} : i = 1, 2, \dots, d(\delta); j = 1, 2, \dots, m(\delta)\}$ form a basis for V_F . In particular, $\dim V_F = m(\delta)l(\delta)$.

Lemma 4.6. *There is a unique function $J_\delta : T^+ \rightarrow \mathcal{M}_{l(\delta) \times l(\delta)}$ such that for all regular point $x = (r, kM)$ in \mathbb{R}^n ,*

$$P^\delta(x) = Y^\delta(kM)J_\delta(r). \quad (4.9)$$

Also for each $r \in T^+$, $J_\delta(r)$ is invertible, and consequently for all regular point $x = (r, kM)$

$$Y^\delta(kM) = P^\delta(x)[J_\delta(r)]^{-1}. \quad (4.10)$$

Proof. First part follows from Lemma 4.3, since $P^\delta \in \mathcal{E}^\delta(\mathbb{R}^n)$. Now, if possible, let for some r_0 in T^+ , $J_\delta(r_0)$ be not invertible i.e $\det(J_\delta(r_0)) = 0$ (where \det stands for the determinant). This implies that the columns of $J_\delta(r_0)$ namely $[J_{1j}(r_0), J_{2j}(r_0), \dots, J_{l(\delta)j}(r_0)]^t$, $1 \leq j \leq l(\delta)$ are linearly dependent as vectors in $\mathbb{C}^{l(\delta)}$. Equating the entries of first row in (4.9) for $x = (r_0, kM)$ we get $(1 \leq j \leq l(\delta))$,

$$P_{1j}^\delta(x) = Y_{11}^\delta(kM)J_{1j}(r_0) + Y_{12}^\delta(kM)J_{2j}(r_0) + \dots + Y_{1l(\delta)}^\delta(kM)J_{l(\delta)j}(r_0).$$

Therefore P_{1j}^δ s are linearly dependent when restricted to the orbit through r_0 and hence by Kostant-Rallis Theorem (Theorem 3.2) P_{1j}^δ s are linearly dependent which is a contradiction. Therefore $J_\delta(r)$ is invertible for all $r \in T^+$. \square

Remark 4.7. (i) Let $x = (r, kM)$ be a regular point. Since $Y^\delta(kM)$ has a left inverse, and $J_\delta(r)$ is invertible $P^\delta(x)$ also has a left inverse.

(ii) The function J_δ is related to Υ_δ (see (4.3)) by

$$[J_\delta(r)]^*[J_\delta(r)] = \Upsilon_\delta(r), \quad \forall r \in T^+. \quad (4.11)$$

(iii) Let $K = SO(n)$. Let $Y^{\delta_m}, P^{\delta_m}$ be as in (4.4) and (4.5). Then we have seen that

$$P^{\delta_m}(x) = r^m Y^{\delta_m}(\omega), \quad x = r\omega, \quad r > 0, \omega \in S^{n-1}.$$

Therefore $J_{\delta_m} : T^+ = (0, \infty) \rightarrow \mathcal{M}_{1 \times 1}$ is given by $J_{\delta_m}(r) = r^m$.

The next proposition follows by using (4.10) in Lemma 4.3, and by (i) in the previous remark.

Proposition 4.8. *Suppose F is in $\mathcal{E}^\delta(\mathbb{R}^n)$. Then there is a unique (on the set of regular points) K -invariant function $G : \mathbb{R}^n \longrightarrow \mathcal{M}_{l(\delta) \times l(\delta)}$ such that for all regular points x (hence for almost every x),*

$$F(x) = P^\delta(x)G(x).$$

Throughout this paper we use the following convention : when we say that a matrix-valued function is a polynomial we mean that each entry of the function is a polynomial.

Corollary 4.9. *Let $F \in \mathcal{E}^\delta(\mathbb{R}^n)$ be a polynomial. Then there is a unique K -invariant polynomial $G : \mathbb{R}^n \longrightarrow \mathcal{M}_{l(\delta) \times l(\delta)}$ such that for all $x \in \mathbb{R}^n$,*

$$F(x) = P^\delta(x)G(x).$$

Proof. Since the set of regular points is dense in \mathbb{R}^n , uniqueness follows from Remark 4.7 (i). Now let G be as in the previous proposition. It is enough to show that each entry of G is equal to a polynomial on the set of regular points. Consider F_{11} . For all regular points x we have

$$F_{11}(x) = \sum_{p=1}^{l(\delta)} P_{1p}^\delta(x) G_{p1}(x). \quad (4.12)$$

By Theorem 3.1 we have

$$S_\delta^\vee = IH_\delta^\vee,$$

where $S_\delta^\vee \subset S$ denotes the space of all polynomials of type δ^\vee . Clearly $F_{11} \in S_\delta^\vee$. Therefore there exists K -invariant polynomials I_{ij} such that for all $x \in \mathbb{R}^n$

$$F_{11}(x) = \sum_{i=1}^{d(\delta)} \sum_{j=1}^{l(\delta)} I_{ij}(x) P_{ij}^\delta(x). \quad (4.13)$$

Since P_{ij}^δ s are linearly independent, by Kostant-Rallis Theorem (Theorem 3.2) so are their restrictions to any regular orbit. Comparing equations (4.12) and (4.13), restricted to a orbit passing through a regular point x , we get $G_{p1}(x) = I_{p1}(x)$ for all $p = 1, 2, \dots, l(\delta)$. Similar proof works for other entries of G . \square

5. HEISENBERG GROUP : REPRESENTATIONS, WEYL TRANSFORM, SPHERICAL FUNCTIONS

The Heisenberg group \mathbb{H}^n is the Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R}$ and group operation

$$(z, t)(z', t') = (z + z', t + t' + \operatorname{Im}(z \cdot \bar{z}')).$$

For the following see Geller [6]. For real non-zero λ , let

$$\mathcal{H}^\lambda = \{u \text{ holomorphic on } \mathbb{C}^n : \int_{\mathbb{C}^n} |u(w)|^2 d\tilde{w}^\lambda = \|u\|^2 < \infty\},$$

where the measure $d\tilde{w}^\lambda$ is given by

$$d\tilde{w}^\lambda = (2|\lambda|/\pi)^n e^{-2|\lambda||w|^2} dw d\bar{w}.$$

The space \mathcal{H}^λ is a Hilbert space and an orthonormal basis is given by $\{u_\nu^\lambda : \nu \in \mathbb{Z}_+^n\}$, where \mathbb{Z}_+^n is the set of non-negative n -tuple, and

$$u_\nu^\lambda(w) = [(2|\lambda|)^{1/2} w]^\nu (\nu!)^{1/2}.$$

(Here $\nu! = \prod_{j=1}^n \nu_j!$ and $w^\nu = \prod_{j=1}^n w_j^{\nu_j}$.) Let $\mathcal{O}(\mathcal{H}^\lambda)$ denote the set of all linear operators in \mathcal{H}^λ whose domain of definition contains $\mathcal{P}(\mathbb{C}^n)$, the space of holomorphic polynomial on \mathbb{C}^n . For $\lambda > 0$, define $\bar{W}_j^\lambda, W_j^\lambda \in \mathcal{O}(\mathcal{H}^\lambda)$ as follows: if $P \in \mathcal{P}(\mathbb{C}^n)$,

$$\bar{W}_j^\lambda P(w) = 2|\lambda| w_j P(w) \text{ and } W_j^\lambda P(w) = \frac{\partial P}{\partial w_j}(w),$$

while if $\lambda < 0$ the situation is reversed (In Geller [6], the notation $W_{j\lambda}^+, W_{j\lambda}$ are used for $\bar{W}_j^\lambda, W_j^\lambda$ respectively). We have the commutation relations

$$[\bar{W}_j^\lambda, W_k^\lambda] = -2\delta_{jk}\lambda I, \quad [W_j^\lambda, W_k^\lambda] = 0, \quad [\bar{W}_j^\lambda, \bar{W}_k^\lambda] = 0, \quad (5.1)$$

where I denote the identity operator. Let $\bar{W}^\lambda = (\bar{W}_1^\lambda, \bar{W}_1^\lambda, \dots, \bar{W}_n^\lambda)$, $W^\lambda = (W_1^\lambda, W_1^\lambda, \dots, W_n^\lambda)$; and for $z \in \mathbb{C}^n$, let $z \cdot \bar{W}^\lambda, z \cdot W^\lambda$ denote the operators $z_1 \bar{W}_1^\lambda + z_2 \bar{W}_2^\lambda + \dots + z_n \bar{W}_n^\lambda$ and $z_1 W_1^\lambda + z_2 W_2^\lambda + \dots + z_n W_n^\lambda$ respectively. Then $i(-z \cdot \bar{W}^\lambda + \bar{z} \cdot W^\lambda)$ being self-adjoint,

$$V_z^\lambda = \exp(-z \cdot \bar{W}^\lambda + \bar{z} \cdot W^\lambda)$$

extends to a unitary operator on \mathcal{H}^λ which satisfy

$$V_z^\lambda V_w^\lambda = \exp(2i\lambda \operatorname{Im}(z \cdot \bar{w})) V_{z+w}^\lambda, \quad (5.2)$$

and has an explicit formulae given as follows: if $u \in \mathcal{H}_\lambda$,

$$\begin{aligned} (V_z^\lambda u)(w) &= u(w + \bar{z}) \exp[-2\lambda(w \cdot z + |z|^2/2)] \text{ for } \lambda > 0 \\ &= u(w - z) \exp[2\lambda(-w \cdot \bar{z} + |z|^2/2)] \text{ for } \lambda < 0. \end{aligned}$$

In view of (5.2) we have a representation Π^λ of \mathbb{H}^n on \mathcal{H}^λ , given by $\Pi^\lambda(z, t) = e^{i\lambda t} V_z^\lambda$. Explicitly Π^λ is given as follows: if $u \in \mathcal{H}^\lambda$,

$$\begin{aligned} (\Pi^\lambda(z, t)u)(w) &= u(w + \bar{z}) \exp[-2\lambda(w \cdot z + |z|^2/2)] e^{i\lambda t} \text{ for } \lambda > 0 \\ &= u(w - z) \exp[2\lambda(-w \cdot \bar{z} + |z|^2/2)] e^{i\lambda t} \text{ for } \lambda < 0. \end{aligned}$$

In fact, these are all the unitary irreducible representations of \mathbb{H}^n which are non-trivial on the center. Note that $\Pi^\lambda(z, 0) = V_z^\lambda$. We will write $\Pi^\lambda(z)$ instead of $\Pi^\lambda(z, 0)$. Since V_z^λ is unitary, we can define a map $\mathcal{G}^\lambda : \mathcal{S}(\mathbb{C}^n) \longrightarrow \mathcal{O}(\mathcal{H}^\lambda)$ by

$$\mathcal{G}^\lambda f = \int_{\mathbb{C}^n} f(z) V_z^\lambda dz d\bar{z} = \int_{\mathbb{C}^n} f(z) \Pi^\lambda(z) dz d\bar{z}.$$

The operator $\mathcal{G}^\lambda f$ is called the Weyl transform of f . Let $\mathcal{S}_2(\mathcal{H}^\lambda)$ stand for the Hilbert space of Hilbert-Schmidt operators on \mathcal{H}^λ with the inner product $\langle T, S \rangle = \operatorname{tr}(TS^*)$. Let $\|\cdot\|_{\text{HS}}$ denote the Hilbert-Schmidt norm. Now we state the Plancherel theorem for Weyl transform.

Theorem 5.1. (Geller [6], Theorem 1.2) *If $f \in \mathcal{S}(\mathbb{C}^n)$, then $\mathcal{G}^\lambda f \in \mathcal{S}_2(\mathcal{H}^\lambda)$ and*

$$\|f\|_2^2 = \pi^{-n} (2|\lambda|)^n \|\mathcal{G}^\lambda f\|_{\text{HS}}^2.$$

The map \mathcal{G}^λ may then be extended as a constant multiple of a unitary map from $L^2(\mathbb{C}^n)$ onto $\mathcal{S}_2(\mathcal{H}^\lambda)$. A polarization of the above formula gives

$$\langle f, g \rangle = \pi^{-n} (2|\lambda|)^n \langle \mathcal{G}^\lambda f, \mathcal{G}^\lambda g \rangle,$$

where $f, g \in L^2(\mathbb{C}^n)$.

For $f, g \in L^2(\mathbb{C}^n)$, define the twisted convolution

$$f \times^\lambda g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{2i\lambda \operatorname{Im}(z \cdot \bar{w})}dw.$$

Then it is well-known that, $f \times^\lambda g \in L^2(\mathbb{C}^n)$ and

$$\mathcal{G}^\lambda(f \times^\lambda g) = \mathcal{G}^\lambda(f)\mathcal{G}^\lambda(g).$$

Next, we extend this definition to a suitable subset of $\mathcal{S}'(\mathbb{C}^n)$, the space of tempered distributions on \mathbb{C}^n (see Geller [6], page 624-625). We say that $T \in \mathcal{S}'(\mathbb{C}^n)$ is Weyl transformable if there exist $R \in \mathcal{O}(\mathcal{H}^\lambda)$ such that

$$T(f) = \pi^{-n}(2|\lambda|)^n \sum_{\nu \in \mathbb{Z}_+^n} (Ru_\nu^\lambda, (\mathcal{G}^\lambda f)u_\nu^\lambda) \quad \forall f \in \mathcal{S}(\mathbb{C}^n),$$

where the series converges absolutely. It is shown in [6] that if such an R exists then it is unique. In this case we call R to be the Weyl transform of T and write $\mathcal{G}^\lambda(T) = R$. It is clear from the polarization of Plancherel Theorem (Theorem 5.1) that this definition agrees with the previous definition of Weyl transform if T is given by an L^2 -function. In the course of proving the uniqueness of R , Geller proved that, if we fix a $\gamma \in \mathbb{Z}_+^n$, then for each $\alpha, \beta \in \mathbb{Z}_+^n$ there exist $f_{\alpha\beta} \in \mathcal{S}(\mathbb{C}^n)$ such that $\mathcal{G}^\lambda(f_{\alpha\beta})u_\gamma^\lambda = \delta_{\alpha\gamma}u_\beta^\lambda$. Taking $\beta = \alpha$, in particular we have the following: Fix $\gamma \in \mathbb{Z}_+^n$. Then for each $\alpha \in \mathbb{Z}_+^n$, there exists $f_\alpha \in \mathcal{S}(\mathbb{C}^n)$ such that $\mathcal{G}^\lambda(f_\alpha)u_\gamma^\lambda = \delta_{\alpha\gamma}u_\alpha^\lambda$. From this fact, the next proposition follows easily.

Proposition 5.2. *Let $\{T_j\}$ be a sequence of tempered distributions which converge to a tempered distribution T in the topology of $\mathcal{S}'(\mathbb{C}^n)$, i.e $T_j(f) \rightarrow T(f)$ for all $f \in \mathcal{S}(\mathbb{C}^n)$. Assume that all T_j 's and T are Weyl transformable. Then for any $u, v \in \mathcal{P}(\mathbb{C}^n)$, $\langle \mathcal{G}^\lambda(T_j)u, v \rangle \rightarrow \langle \mathcal{G}^\lambda(T)u, v \rangle$.*

Define $\mathcal{F} : \mathcal{S}(\mathbb{C}^n) \rightarrow \mathcal{S}(\mathbb{C}^n)$ by

$$(\mathcal{F}f)(\zeta) = \int_{\mathbb{C}^n} \exp(-z \cdot \bar{\zeta} + \bar{z} \cdot \zeta) f(z) dz d\bar{z}.$$

This is a modification of the usual Euclidean Fourier transform \mathcal{F} , the relation being that $(\mathcal{F}'f)(\zeta) = (\mathcal{F}f)(-2i\zeta)$. So we can extend \mathcal{F}' as a continuous, linear, one-to-one mapping of $\mathcal{S}'(\mathbb{C}^n)$ onto $\mathcal{S}'(\mathbb{C}^n)$. Let $T \in \mathcal{S}'(\mathbb{C}^n)$ be such that $\mathcal{F}'^{-1}T$ is Weyl transformable. Then we define the Weyl correspondence \mathcal{W}^λ of T by

$$\mathcal{W}^\lambda(T) = \mathcal{G}^\lambda(\mathcal{F}'^{-1}T).$$

On \mathbb{H}^n , the differential operators

$$T = \frac{\partial}{\partial t}, \quad Z_j = \frac{\partial}{\partial \bar{z}_j} - iz_j \frac{\partial}{\partial t}, \quad \bar{Z}_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}$$

are the left invariant vector fields corresponding to the one parameter family of subgroups $\Gamma_0 = \{(0, s) : s \in \mathbb{R}\}$, $\Gamma_j = \{se_j, 0 : s \in \mathbb{R}\}$ and $\bar{\Gamma}_j = \{s\bar{e}_j, 0 : s \in \mathbb{R}\}$ respectively, where $\{e_1, e_2, \dots, e_n\}$ be the usual basis for \mathbb{C}^n . In [6], page-651, the notation differ slightly. Geller uses \bar{Z}_j for our operator Z_j (and Z_j for \bar{Z}_j). These form a basis for $\mathcal{L}(\mathfrak{h}_n)$, the set of all left invariant differential operators on \mathbb{H}^n . Here \mathfrak{h}_n is the Lie algebra of \mathbb{H}^n . For each $D \in \mathcal{L}(\mathfrak{h}_n)$, let D^λ denote the operator on \mathbb{C}^n obtained by replacing each copy of $\partial/\partial t$ in D by $-i\lambda$. Define

$$\mathcal{L}^\lambda(\mathbb{C}^n) = \{D^\lambda : D \in \mathcal{L}(\mathfrak{h}_n)\}, \text{ and } \mathcal{R}^\lambda(\mathbb{C}^n) = \{D^{-\lambda} : D \in \mathcal{L}(\mathfrak{h}_n)\}.$$

Then

$$L_j^\lambda = \frac{\partial}{\partial \bar{z}_j} - \lambda z_j, \quad \bar{L}_j^\lambda = \frac{\partial}{\partial z_j} + \lambda \bar{z}_j$$

form a basis for $\mathcal{L}^\lambda(\mathbb{C}^n)$, and

$$R_j^\lambda = \frac{\partial}{\partial \bar{z}_j} + \lambda z_j, \quad \bar{R}_j^\lambda = \frac{\partial}{\partial z_j} - \lambda \bar{z}_j$$

form a basis for $\mathcal{R}^\lambda(\mathbb{C}^n)$. In [6], page-619, these are denoted by $\widetilde{\mathcal{Z}}_{j\lambda}$, $\mathcal{Z}_{j\lambda}$, $\widetilde{\mathcal{Z}}_{j\lambda}^R$, $\mathcal{Z}_{j\lambda}^R$ respectively. Note that the action of Z_j and \bar{Z}_j on a function of the form $e^{-i\lambda t}f(z)$ are given by

$$Z_j(e^{-i\lambda t}f) = e^{-i\lambda t}L_j^\lambda(f), \quad \bar{Z}_j(e^{-i\lambda t}f) = e^{-i\lambda t}\bar{L}_j^\lambda(f).$$

We also have the commutation relations

$$[\bar{L}_j^\lambda, L_k^\lambda] = -2\delta_{jk}\lambda I, \quad [L_j^\lambda, L_k^\lambda] = 0, \quad [\bar{L}_j^\lambda, \bar{L}_k^\lambda] = 0. \quad (5.3)$$

$$[\bar{R}_j^\lambda, R_k^\lambda] = 2\delta_{jk}\lambda I, \quad [R_j^\lambda, R_k^\lambda] = 0, \quad [\bar{R}_j^\lambda, \bar{R}_k^\lambda] = 0. \quad (5.4)$$

The following proposition tells how the operators $L_j^\lambda, \bar{L}_j^\lambda, R_j^\lambda, \bar{R}_j^\lambda$ behave under \mathcal{G}^λ . The proof can be found in [6], page 624-625.

Proposition 5.3. *If $T \in \mathcal{S}'(\mathbb{C}^n)$ is Weyl transformable, then so are $L_j^\lambda T, \bar{L}_j^\lambda T, R_j^\lambda T, \bar{R}_j^\lambda T$. Also*

$$\begin{aligned} \mathcal{G}^\lambda(L_j^\lambda T) &= -\mathcal{G}^\lambda(T)W_j^\lambda, \quad \mathcal{G}^\lambda(\bar{L}_j^\lambda T) = \mathcal{G}^\lambda(T)\bar{W}_j^\lambda, \\ \mathcal{G}^\lambda(R_j^\lambda T) &= -W_j^\lambda\mathcal{G}^\lambda(T), \quad \mathcal{G}^\lambda(\bar{R}_j^\lambda T) = \bar{W}_j^\lambda\mathcal{G}^\lambda(T). \end{aligned}$$

Let δ_0 denote the Dirac delta distribution at origin. Since $\mathcal{G}^\lambda(\delta_0)$ is the identity operator, from the above proposition we get the following corollary. We write $\mathcal{G}^\lambda(D)$ for $\mathcal{G}^\lambda(D\delta_0)$, if $D \in \mathcal{L}^\lambda(\mathbb{C}^n)$ or $\mathcal{R}^\lambda(\mathbb{C}^n)$.

Corollary 5.4. *Let T be a Weyl transformable tempered distribution. Then $\mathcal{G}^\lambda(DT) = \mathcal{G}^\lambda(T)\mathcal{G}^\lambda(D)$ if $D \in \mathcal{L}^\lambda(\mathbb{C}^n)$; and $\mathcal{G}^\lambda(DT) = \mathcal{G}^\lambda(D)\mathcal{G}^\lambda(T)$ if $D \in \mathcal{R}^\lambda(\mathbb{C}^n)$.*

Let $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)$ denote the space of all polynomials on the underlying real vector space $\mathbb{C}_{\mathbb{R}}^n$ of \mathbb{C}^n . Clearly $\mathbb{C}_{\mathbb{R}}^n$ can be identified with \mathbb{R}^{2n} . In other words the elements of $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)$ are polynomials in z and \bar{z} with complex coefficients. From now on we use the following convention : when we write “**polynomial**”, we mean a polynomial in z and \bar{z} , i.e. an element in $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)$, and elements of $\mathcal{P}(\mathbb{C}^n)$ are called “**holomorphic polynomials**” i.e. polynomials in z only. For a monomial $p(\zeta) = \zeta^\rho \bar{\zeta}^\gamma$ (ρ, γ multi-indices), we set

$$\theta_1^\lambda(p) = (\bar{R}^\lambda)^\gamma (-R^\lambda)^\rho, \quad \theta_2^\lambda(p) = (-R^\lambda)^\rho (\bar{R}^\lambda)^\gamma$$

and

$$\tau_1^\lambda(p) = (\bar{W}^\lambda)^\gamma (W^\lambda)^\rho, \quad \tau_2^\lambda(p) = (W^\lambda)^\rho (\bar{W}^\lambda)^\gamma.$$

In the above $(\bar{R}^\lambda)^\gamma = (\bar{R}_1^\lambda)^{\gamma_1} \cdots (\bar{R}_n^\lambda)^{\gamma_n}$ (order does not matter because of commutation relations (5.4)), where $\gamma = (\gamma_1, \dots, \gamma_n)$. The other expressions are similarly

defined. Define

$$\theta^\lambda(p) = \frac{1}{2}(\theta_1^\lambda(p) + \theta_2^\lambda(p)), \quad \tau^\lambda(p) = \frac{1}{2}(\tau_1^\lambda(p) + \tau_2^\lambda(p)).$$

We extend them to all polynomials by linearity. Note that by Proposition 5.3,

$$\mathcal{G}^\lambda(\theta_1^\lambda(p)) = \tau_1^\lambda(p), \quad \mathcal{G}^\lambda(\theta_2^\lambda(p)) = \tau_2^\lambda(p), \quad \mathcal{G}^\lambda(\theta^\lambda(p)) = \tau^\lambda(p), \quad (5.5)$$

for any polynomial p .

Proposition 5.5. (Geller [6], Proposition 2.1 (a), 2.7)

- (a) *If p is a polynomial, then $\mathcal{F}^{\prime-1}p$ is Weyl transformable and hence $\mathcal{W}^\lambda(p)$ is well defined.*
- (b) *If p is a $U(n)$ -harmonic polynomial, then $\mathcal{W}^\lambda(p) = \tau_1^\lambda(p) = \tau_2^\lambda(p) = \tau^\lambda(p)$.*

Remark 5.6. In fact one can prove that for any polynomial p , $\mathcal{W}^\lambda(p) = \tau^\lambda(p)$. Since we will be dealing with only harmonic polynomials we don't need this general result.

We conclude this section with a short discussion about Gelfand pairs and K -spherical functions on \mathbb{H}^n . For details see Benson et al. [2]. Let K be a compact Lie subgroup of $\text{Aut}(\mathbb{H}^n)$, the group automorphisms of \mathbb{H}^n . Each $k \in U(n)$, the group of $n \times n$ unitary matrices on \mathbb{C}^n , gives rise to an automorphism of \mathbb{H}^n , via $k \cdot (z, t) = (k \cdot z, t)$. So we can consider $U(n)$ as a subgroup of $\text{Aut}(\mathbb{H}^n)$. In fact $U(n)$ is a maximal connected, compact subgroup of $\text{Aut}(\mathbb{H}^n)$, and thus any connected, compact subgroup of $\text{Aut}(\mathbb{H}^n)$ is the conjugate of a subgroup K of $U(n)$. The pair (K, \mathbb{H}^n) is called a Gelfand pair if $L_K^1(\mathbb{H}^n)$, the convolution subalgebra of K -invariant L^1 functions on \mathbb{H}^n , is commutative. Since conjugates of K form Gelfand pairs with \mathbb{H}^n if and only if K does, and produce the same joint eigenfunctions for all $D \in \mathcal{L}_K(\mathfrak{h}_n)$ (the set of all differential operators on \mathbb{H}^n that are invariant under the action of K and the left action of \mathbb{H}^n), which is our main interest in this paper, we will always assume that we are dealing with a connected, compact subgroup K of $U(n)$. The K -action on \mathbb{C}^n gives rise to a natural action on a function f on \mathbb{C}^n

given by $k \cdot f(z) = f(k^{-1} \cdot z)$. Under this action we have the decomposition of $\mathcal{P}(\mathbb{C}^n)$ into K -irreducible subspaces as

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{\alpha \in \Lambda} V_\alpha \text{ (algebraic direct sum).}$$

Here Λ denotes a countably infinite index set. Since $\mathcal{P}_m(\mathbb{C}^n)$, the space of homogeneous holomorphic polynomials of degree m , is invariant under the K -action (as $K \subset U(n)$), we can take each V_α to be contained in some $\mathcal{P}_m(\mathbb{C}^n)$. Define the unitary representation U^λ of K on the Hilbert space \mathcal{H}^λ as follows: if $u \in \mathcal{H}^\lambda$,

$$U^\lambda(k)u = \begin{cases} \bar{k} \cdot u & \text{if } \lambda > 0 \\ k \cdot u & \text{if } \lambda < 0. \end{cases}$$

Since (K, \mathbb{H}^n) is a Gelfand pair, U^λ is multiplicity free (see Benson et al. [2], Theorem 1.7). $\mathcal{P}(\mathbb{C}^n)$ being dense in \mathcal{H}^λ , we get the same decomposition of \mathcal{H}^λ into U^λ -irreducible subspaces :

$$\mathcal{H}^\lambda = \bigoplus_{\alpha \in \Lambda}^\perp V_\alpha \text{ (orthogonal Hilbert space decomposition).}$$

Choose a basis $\{e_{\alpha\nu}^\lambda : \nu = 1, 2, \dots, d(\alpha)\}$ for each V_α so that $\{e_{\alpha\nu}^\lambda : \alpha \in \Lambda, \nu = 1, 2, \dots, d(\alpha)\}$ is an orthonormal basis for \mathcal{H}^λ . We will use this basis in the later sections. The behaviour of K -action on a function under Weyl transform is given by the following proposition.

Proposition 5.7. (Geller [6], Proposition 1.3)

- (a) $\Pi^\lambda(k \cdot z) = (U^\lambda(k))^{-1} \Pi^\lambda(z) (U^\lambda(k))$.
- (b) If $f \in L^2(\mathbb{C}^n)$, $\mathcal{G}^\lambda(k \cdot f) = (U^\lambda(k)) \mathcal{G}^\lambda f (U^\lambda(k))^{-1}$.
- (c) For any polynomial p , $\mathcal{W}^\lambda(k \cdot p) = (U^\lambda(k)) \mathcal{W}^\lambda(p) (U^\lambda(k))^{-1}$.

In fact, (c) is not proved in [6]. But using the definition of Weyl transform of a tempered distribution, one can show that (b) is true for any Weyl transformable tempered distribution. Since Euclidean Fourier transform commutes with the action of K , (c) follows.

A smooth K -invariant function $\phi : \mathbb{H}^n \rightarrow \mathbb{C}$ is called K -spherical if $\phi(0, 0) = 1$ and ϕ is a joint eigenfunction for all $D \in \mathcal{L}_K(\mathfrak{h}_n)$. In [2], the authors describe all bounded K -spherical functions, their forms, and the corresponding eigenvalues. We summarise these in the following theorem. Assume that dk is the normalized Haar measure on K .

Theorem 5.8. *There are two distinct classes of bounded K -spherical functions.*

(a) *The first type is parametrized by $(\lambda, \alpha) \in \mathbb{R}^* \times \Lambda$ (\mathbb{R}^* denotes the set of all non-zero real numbers), and given by*

$$\phi_\alpha^\lambda(z, t) = \int_K \langle \Pi^\lambda(k \cdot (z, t)) v, v \rangle dk,$$

for any unit vector $v \in V_\alpha$. Each ϕ_α^λ has the form

$$\phi_\alpha^\lambda(z, t) = e^{i\lambda t} q_\alpha^\lambda(z) e^{-|\lambda||z|^2},$$

where $q_\alpha^\lambda(z)$ is a polynomial. The corresponding eigenvalue $\tilde{\mu}_\alpha^\lambda$ s are distinct, and can be obtained from the equation (for any non-zero $v \in V_\alpha$),

$$\Pi^\lambda(D)v = \tilde{\mu}_\alpha^\lambda(D)v \quad \forall D \in \mathcal{L}_K(\mathfrak{h}_n). \quad (5.6)$$

(b) *The second type is parametrized by \mathbb{C}^n/K , the space of K -orbits in \mathbb{C}^n . For $\omega \in \mathbb{C}^n$ we write η_ω for the associated K -spherical function. One has $\eta_\omega = \eta_{\omega'}$, if $K \cdot \omega = K \cdot \omega'$. $\eta_\omega(z, t)$ is independent of t , and is given by*

$$\eta_\omega(z, t) = \int_K e^{i\operatorname{Re}(\omega, k \cdot z)} dk.$$

Let $\mathcal{L}_K^\lambda(\mathbb{C}^n) = \{D^\lambda : D \in \mathcal{L}_K(\mathfrak{h}_n)\}$. Clearly $\mathcal{L}_K^\lambda(\mathbb{C}^n) \subset \mathcal{L}^\lambda(\mathbb{C}^n)$. Define

$$\psi_\alpha^\lambda(z) = \frac{1}{\kappa_\alpha^\lambda} \int_K \langle \Pi^\lambda(k \cdot z) v, v \rangle dk,$$

where v is any unit vector in V_α , and κ_α^λ is the square of L^2 norm of $\int_K \langle \Pi^\lambda(k \cdot z) v, v \rangle dk$.

The functions ψ_α^λ are real valued and $\psi_\alpha^\lambda = \psi_{\alpha^{-\lambda}}^{-\lambda}$ (see [2], Remark, page-428). Therefore we can write $\phi_\alpha^{-\lambda}(z, t) = \kappa_\alpha^\lambda e^{-i\lambda t} \psi_\alpha^\lambda$. Then with the property $\|\psi_\alpha^\lambda\|_2^2 = \frac{1}{\kappa_\alpha^\lambda}$, $\psi_\alpha^\lambda(z)$ is the unique (upto constant multiple) bounded joint eigenfunction for all $D^\lambda \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with the eigenvalue μ_α^λ , where $\mu_\alpha^\lambda(D^\lambda) = \tilde{\mu}_\alpha^{-\lambda}(D)$ for all $D \in \mathcal{L}_K(\mathfrak{h}_n)$.

Remark 5.9. Equation (5.6) can be restated in terms of Weyl transform as (for any non-zero $v \in V_\alpha$)

$$\mathcal{G}^\lambda(D)v = \mu_\alpha^\lambda(D)v \quad \forall D \in \mathcal{L}_K^\lambda(\mathbb{C}^n).$$

Remark 5.10. Let $K = U(n)$. $\mathcal{L}_K^\lambda(\mathbb{C}^n)$ is generated by the special Hermite operator

$$\mathcal{L}^\lambda := \sum_{j=1}^n L_j^\lambda \bar{L}_j^\lambda + \bar{L}_j^\lambda L_j^\lambda = \sum_{j=1}^n \frac{\partial}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} - \lambda |z|^2 + \sum_{j=1}^n \left(\bar{z}_j \frac{\partial}{\partial \bar{z}_j} - z_j \frac{\partial}{\partial z_j} \right).$$

The decomposition of $\mathcal{P}(\mathbb{C}^n)$ into K -irreducible subspaces is given by $\mathcal{P}(\mathbb{C}^n) = \bigoplus_{k \in \mathbb{Z}^+} \mathcal{P}_k(\mathbb{C}^n)$. Recall that $\mathcal{P}_k(\mathbb{C}^n)$ is the space of all homogeneous holomorphic polynomial on \mathbb{C}^n of degree k . The bounded K -spherical functions are parametrized by \mathbb{Z}^+ , the set of non negative integers. The corresponding ψ_k^λ 's are given by

$$\psi_k^\lambda(z) = \pi^{-n} (2|\lambda|)^n L_k^{n-1}(2|\lambda||z|^2) e^{-|\lambda||z|^2},$$

where L_k^{n-1} is the Laguerre polynomial of type $n-1$, and the corresponding eigenvalues are given by $\mu_k^\lambda(\mathcal{L}^\lambda) = -2|\lambda|(2k+n)$. It is easy to see that (or by Corollary 2.3 in [2]), $\psi_k^\lambda = \Sigma_{V_\alpha \subset \mathcal{P}_k(\mathbb{C}^n)} \psi_\alpha^\lambda$.

6. WEYL TRANSFORM OF K -INVARIANT FUNCTIONS

Through out this section we assume that (K, \mathbb{H}^n) is a Gelfand pair. Let $\lambda \in \mathbb{R}^*$ be fixed.

Proposition 6.1. *Let $T \in \mathcal{S}'(\mathbb{C}^n)$ be K -invariant and Weyl transformable. Then $\mathcal{G}^\lambda T$ is a constant multiple of the identity operator on each V_α .*

Proof. For simplicity of notation we suppress the superscript λ from the notation introduced in the previous section. Since T is K -invariant, there exists a sequence $\{f_j\}$ of smooth, compactly supported, K -invariant functions on \mathbb{C}^n such that f_j converge to T in the topology of $\mathcal{S}'(\mathbb{C}^n)$. By Proposition 5.7 (b), each $\mathcal{G}f_j$ commutes with all $U(k)$. Since the representation U of K on the various V_α 's are irreducible and inequivalent, $\mathcal{G}f_j$ preserves each V_α . Thus, by Schur's Lemma, $\mathcal{G}f_j$ is constant on each V_α . Hence by Proposition 5.2 we are done. \square

Since the Euclidean Fourier transform commutes with the action of K , an easy consequence of the above proposition and Proposition 4.5 (a) is the following corollary.

Corollary 6.2. *Weyl correspondence of a K -invariant polynomial is constant on each V_α .*

Proposition 6.3. $\mathcal{G}^\lambda(\psi_\alpha^\lambda) = \mathcal{P}_\alpha$, where \mathcal{P}_α denotes the projection operator onto V_α .

Proof. As usual we suppress the superscript λ . By the previous proposition, $\mathcal{G}(\psi_\alpha)$ is constant on each V_β , say $c_\beta I$. Let $v \in V_\beta$ be non-zero. Then

$$\mathcal{G}(\psi_\alpha)v = c_\beta v.$$

Let $D \in \mathcal{L}_K(\mathbb{C}^n)$. By Corollary 5.4 we have

$$\mathcal{G}(\psi_\alpha)\mathcal{G}(D)v = \mathcal{G}(D\psi_\alpha)v = \mu_\alpha(D)\mathcal{G}(\psi_\alpha)v = c_\beta\mu_\alpha(D)v.$$

Again, by Remark 5.9,

$$\mathcal{G}(\psi_\alpha)\mathcal{G}(D)v = \mu_\beta(D)\mathcal{G}(\psi_\alpha)v = c_\beta\mu_\beta(D)v.$$

Therefore we have

$$c_\beta\mu_\alpha(D)v = c_\beta\mu_\beta(D)v.$$

This is true for all $D \in \mathcal{L}_K(\mathbb{C}^n)$. Since $\mu_\beta \neq \mu_\alpha$ for $\beta \neq \alpha$, we get $c_\beta = 0$. Therefore $\mathcal{G}(\psi_\alpha)$ is zero on V_β , if $\beta \neq \alpha$. Now take a unit vector u from V_α . Then

$$\begin{aligned} c_\alpha &= \langle \mathcal{G}(\psi_\alpha)u, u \rangle \\ &= \int_{\mathbb{C}^n} \psi_\alpha(z) \langle \Pi(z)u, u \rangle dz \\ &= \int_K \int_{\mathbb{C}^n} \psi_\alpha(z) \langle \Pi(k \cdot z)u, u \rangle dz dk. \end{aligned}$$

Since Π is unitary, $|\langle \Pi(k \cdot z)u, u \rangle| \leq 1$. Therefore we can use Fubini's theorem to interchange the integrals and the fact that ψ_α are real valued to get

$$c_\alpha = \kappa_\alpha \int_{\mathbb{C}^n} \psi_\alpha(z) \psi_\alpha(z) dz = \kappa_\alpha \|\psi_\alpha\|_2^2 = 1.$$

Hence the proof. □

Proposition 6.4. *Let $f \in L^2(\mathbb{C}^n)$. Then $f = \sum_{\alpha \in \Lambda} f \times^\lambda \psi_\alpha^\lambda$, where the series converges in $L^2(\mathbb{C}^n)$.*

Proof. Since the index set Λ is countable, we can identify Λ with the set of natural numbers \mathbb{N} . For $j \in \mathbb{N}$,

$$\mathcal{G}(f)|_{V_j} = \mathcal{G}(f)\mathcal{P}_j = \mathcal{G}(f)\mathcal{G}(\psi_j) = \mathcal{G}(f \times \psi_j).$$

Therefore,

$$\left\| \mathcal{G}(f) - \mathcal{G}\left(\sum_{j=1}^N f \times \psi_j\right) \right\|_{\text{HS}}^2 = \sum_{j>N} \sum_{\nu=1}^{d(j)} \|\mathcal{G}(f)e_{j\nu}\|_2^2 \longrightarrow 0$$

as $N \longrightarrow \infty$, since $\mathcal{G}(f)$ is a Hilbert-Schmidt operator. Hence by the Plancherel theorem (Theorem 5.1) we are done. \square

The above proposition was also proved in [14].

7. GENERALIZED SPHERICAL FUNCTIONS AND WEYL TRANSFORM OF K-TYPE FUNCTIONS

From now on we assume that (K, \mathbb{H}^n) is a Gelfand pair, where K is a connected, compact subgroup of $U(n)$, whose action on \mathbb{C}^n is polar. More precisely, if we identify $U(n)$ as a subgroup of $SO(2n)$ and \mathbb{C}^n with \mathbb{R}^{2n} , then the action of $K \subset SO(2n)$ on \mathbb{R}^{2n} is polar, so that we can use all the results about polar actions from the first three sections. Our main aims are to find all generalized K -spherical functions (Theorem 7.14) and give a formulae for Weyl transform of a function $F \in \mathcal{S}^\delta(\mathbb{C}^n)$ (Theorem 7.4). Here $\mathcal{S}^\delta(\mathbb{C}^n) := \{F \in \mathcal{E}^\delta(\mathbb{C}^n) : F_{ij} \in \mathcal{S}(\mathbb{C}^n)\}$. Theorem 7.4 can be thought of as a generalization of the Theorem 4.2 in [6], which is a Hecke-Bochner type identity. To prove his theorem, Geller introduced certain Hilbert spaces of linear operators which turned out to be analogous to $L^2(S^{2n-1})$ and showed that the Weyl correspondence of $U(n)$ -harmonic polynomials are dense in these Hilbert spaces. We show that a similar result holds for any K (Proposition 7.8), and use this to prove our theorems.

For two positive integers p and q , let $\mathcal{R}_{p \times q}^\lambda(\mathbb{C}^n)$ denote the set of all $p \times q$ matrices whose entries belong to $\mathcal{R}^\lambda(\mathbb{C}^n)$; $\mathcal{O}_{p \times q}(\mathcal{H}^\lambda)$ denote the set of all $p \times q$ matrices whose entries belong to $\mathcal{O}(\mathcal{H}^\lambda)$; and $\mathcal{H}_{p \times q}^\lambda$ denote the same whose entries belong to \mathcal{H}^λ . For $\mathcal{T} \in \mathcal{O}_{p \times q}(\mathcal{H}^\lambda)$, define its action as follows: if $u \in \mathcal{P}(\mathbb{C}^n)$, the (i, j) th entry of $\mathcal{T}u$ is equal to $\mathcal{T}_{ij}u$. Let $\mathcal{S}'_{p \times q}(\mathbb{C}^n)$ denote the set of all $p \times q$ matrices with entries from $\mathcal{S}'(\mathbb{C}^n)$. For $g \in \mathcal{S}'(\mathbb{C}^n)$ and $F \in \mathcal{S}'_{p \times q}(\mathbb{C}^n)$, we define the following whenever they make sense. For a differential operator D on \mathbb{C}^n , define $DF : \mathbb{C}^n \rightarrow \mathcal{M}_{p \times q}$, by $(DF)_{ij}(z) = DF_{ij}(z)$. If $\mathcal{D} \in \mathcal{R}_{p \times q}^\lambda(\mathbb{C}^n)$, define $\mathcal{D}g : \mathbb{C}^n \rightarrow \mathcal{M}_{p \times q}$, by $(\mathcal{D}g)_{ij}(z) = \mathcal{D}_{ij}g(z)$. Define $\mathcal{G}^\lambda F, \mathcal{W}^\lambda F \in \mathcal{O}_{p \times q}(\mathcal{H}^\lambda)$; $\tau^\lambda(P^\delta) \in \mathcal{O}_{d(\delta) \times l(\delta)}(\mathcal{H}^\lambda)$; and $\theta^\lambda(P^\delta) \in \mathcal{R}_{p \times q}^\lambda(\mathbb{C}^n)$, $\mathcal{F}^{-1}P^\delta \in \mathcal{S}'_{d(\delta) \times l(\delta)}(\mathbb{C}^n)$ similarly. For $S \in \mathcal{O}(H^\lambda)$, $\mathcal{T} \in \mathcal{O}_{p \times q}(\mathcal{H}^\lambda)$, define $\mathcal{T}S \in \mathcal{O}_{p \times q}(\mathcal{H}^\lambda)$ by $(\mathcal{T}S)_{ij} = \mathcal{T}_{ij} \circ S$. Similarly define $S\mathcal{T} \in \mathcal{O}_{p \times q}(\mathcal{H}^\lambda)$. For a $r \times p$ constant matrix C , define $C\mathcal{T} \in \mathcal{O}_{r \times q}(\mathcal{H}^\lambda)$, by $(C\mathcal{T})_{ij} = \sum_{k=1}^p C_{ik} \mathcal{T}_{kj}$.

If f is a joint eigendistribution of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$, then K being a subgroup of $U(n)$, it is also a joint eigendistribution of all $D \in \mathcal{L}_{U(n)}^\lambda(\mathbb{C}^n)$. But $\mathcal{L}_{U(n)}^\lambda(\mathbb{C}^n)$ is generated by the special Hermite operator \mathcal{L}^λ , which is elliptic (see Remark 5.10). So we can assume that f is smooth. Therefore, we will consider only smooth joint eigenfunctions for $\mathcal{L}_K^\lambda(\mathbb{C}^n)$.

Definition 7.1. A function $\Psi \in \mathcal{E}^\delta(\mathbb{C}^n)$ is said to be a generalized K -spherical function of type δ corresponding to μ_α^λ , if it is a joint eigenfunction for all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue μ_α^λ , i.e $D\Psi = \mu_\alpha^\lambda(D)\Psi$ for all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$.

By Proposition 5.5 (b) it follows that, for any K -harmonic (hence $U(n)$ -harmonic) polynomial p , $\theta_1^\lambda(p) = \theta_2^\lambda(p)$. Hence $\theta^\lambda(P^\delta) = \theta_1^\lambda(P^\delta) = \theta_2^\lambda(P^\delta)$. Define $\Psi_\alpha^{\delta, \lambda} \in \mathcal{E}^\delta(\mathbb{C}^n)$, by

$$\Psi_\alpha^{\delta, \lambda} = \theta^\lambda(P^\delta) \psi_\alpha^\lambda.$$

Proposition 7.2. $\Psi_\alpha^{\delta, \lambda}$ is a generalized K -spherical function of type δ corresponding to μ_α^λ .

Proof. We suppress the superscript λ . Since $\theta(P^\delta) \in \mathcal{R}_{d(\delta) \times l(\delta)}(\mathbb{C}^n)$, by Corollary 5.4,

$$\mathcal{G}(\Psi_\alpha^\delta) = \mathcal{G}(\theta(P^\delta))\mathcal{G}(\psi_\alpha) = \tau(P^\delta)\mathcal{G}(\psi_\alpha) = \mathcal{W}(P^\delta)\mathcal{G}(\psi_\alpha).$$

Therefore, by Proposition 5.7, for $k \in K$,

$$\begin{aligned} \mathcal{G}(k^{-1} \cdot \Psi_\alpha^\delta) &= U(k)^{-1}\mathcal{G}(\Psi_\alpha^\delta)U(k) = U(k)^{-1}\mathcal{W}(P^\delta)\mathcal{G}(\psi_\alpha)U(k) \\ &= U(k)^{-1}\mathcal{W}(P^\delta)U(k)U(k)^{-1}\mathcal{G}(\psi_\alpha)U(k) = \mathcal{W}(k^{-1} \cdot P^\delta)\mathcal{G}(k^{-1} \cdot \psi_\alpha) \\ &= \delta(k)\mathcal{W}(P^\delta)\mathcal{G}(\psi_\alpha) = \delta(k)\mathcal{G}(\Psi_\alpha^\delta). \end{aligned}$$

So we get $\Psi_\alpha^\delta(k \cdot z) = \delta(k)\Psi_\alpha^\delta(z)$, and hence $\Psi_\alpha^\delta \in \mathcal{E}^\delta(\mathbb{C}^n)$. Again, $D\Psi_\alpha^\delta = \mu_\alpha(D)\Psi_\alpha^\delta$ for all $D \in \mathcal{L}_K(\mathbb{C}^n)$, because

$$\begin{aligned} \mathcal{G}(D\Psi_\alpha^\delta) &= \mathcal{G}(\Psi_\alpha^\delta)\mathcal{G}(D) = \mathcal{G}(\theta(P^\delta))\mathcal{G}(\psi_\alpha)\mathcal{G}(D) \\ &= \mathcal{G}(\theta(P^\delta))\mathcal{G}(D\psi_\alpha) = \mu_\alpha(D)\mathcal{G}(\theta(P^\delta))\mathcal{G}(\psi_\alpha) = \mu_\alpha(D)\mathcal{G}(\Psi_\alpha^\delta). \end{aligned}$$

Therefore Ψ_α^δ is a generalized K -spherical function of type δ corresponding to μ_α . \square

Note that $\Psi_\alpha^{\delta,\lambda}(z)$ is equal to $e^{-|\lambda||z|^2}$ times a polynomial in $\mathcal{E}^\delta(\mathbb{C}^n)$, and hence by Corollary 4.9, there is a unique K -invariant polynomial $L_\alpha^{\delta,\lambda} : \mathbb{C}^n \rightarrow \mathcal{M}_{l(\delta) \times l(\delta)}$ such that

$$\Psi_\alpha^{\delta,\lambda}(z) = P^\delta(z)L_\alpha^{\delta,\lambda}(z)e^{-|\lambda||z|^2}. \quad (7.1)$$

Define the $l(\delta) \times l(\delta)$ constant matrix $A_\alpha^{\delta,\lambda}$ by

$$\begin{aligned} A_\alpha^{\delta,\lambda} &= \int_{\mathbb{C}^n} [\Psi_\alpha^{\delta,\lambda}(z)]^* \Psi_\alpha^{\delta,\lambda}(z) dz \\ &= \int_{\mathbb{C}^n} [L_\alpha^{\delta,\lambda}(z)]^* \Upsilon_\delta(z) L_\alpha^{\delta,\lambda}(z) e^{-2|\lambda||z|^2} dz. \end{aligned}$$

Clearly $A_\alpha^{\delta,\lambda}$ is positive definite. Let $\alpha(\delta)$ denote the number of linearly independent columns in $\Psi_\alpha^{\delta,\lambda}$. Let C_i denote the i th column. Choose $C_{l(1)}, C_{l(2)}, \dots, C_{l(\alpha(\delta))}$ linearly independent and $l(1) < l(2), \dots < l(\alpha(\delta))$. Let the remaining columns be $C_{m(1)}, C_{m(2)}, \dots, C_{m(l(\delta)-\alpha(\delta))}$, with $m(1) < m(2) < \dots < m(l(\delta) - \alpha(\delta))$. Let

$$I_1 = \{l(1), l(2), \dots, l(\alpha(\delta))\},$$

$$I_2 = \{m(1), m(2), \dots, m(l(\delta) - \alpha(\delta))\}.$$

Then I_1 and I_2 are disjoint, and

$$I_1 \cup I_2 = \{1, 2, \dots, l(\delta)\}.$$

Let $\tilde{\Psi}_\alpha^{\delta, \lambda}$ be the $d(\delta) \times \alpha(\delta)$ matrix whose r th column is $C_{l(r)}$, where $l(r) \in I_1$; and $\tilde{L}_\alpha^{\delta, \lambda}$ be the $l(\delta) \times \alpha(\delta)$ matrix whose r th column is $C_{l(r)}$, where $l(r) \in I_1$. Then clearly we have

$$\tilde{\Psi}_\alpha^{\delta, \lambda}(z) = P^\delta(z) \tilde{L}_\alpha^{\delta, \lambda}(z) e^{-|\lambda||z|^2}.$$

Define

$$\begin{aligned} \tilde{A}_\alpha^{\delta, \lambda} &= \int_{\mathbb{C}^n} [\tilde{\Psi}_\alpha^{\delta, \lambda}(z)]^* \tilde{\Psi}_\alpha^{\delta, \lambda}(z) dz \\ &= \int_{\mathbb{C}^n} [\tilde{L}_\alpha^{\delta, \lambda}(z)]^* \Upsilon_\delta(z) \tilde{L}_\alpha^{\delta, \lambda}(z) e^{-2|\lambda||z|^2} dz. \end{aligned}$$

Note that $\tilde{A}_\alpha^{\delta, \lambda}$ is precisely the $\alpha(\delta) \times \alpha(\delta)$ matrix, obtained by deleting the $m(r)$ th rows and columns from $A_\alpha^{\delta, \lambda}$, where $m(r) \in I_2$.

Lemma 7.3. *If $\alpha(\delta) > 0$, $\tilde{A}_\alpha^{\delta, \lambda}$ is invertible.*

Proof. As usual we suppress the superscript λ . If \tilde{A}_α^δ is not invertible, then there exist a non-zero vector $\underline{e} \in \mathbb{C}^{\alpha(\delta)}$, such that $\langle (\tilde{A}_\alpha^\delta \underline{e}), \underline{e} \rangle = 0$ (here \langle, \rangle denotes the usual hermitian inner product on $\mathbb{C}^{\alpha(\delta)}$). Since

$$\tilde{A}_\alpha^\delta = \int_{\mathbb{C}^n} [\tilde{\Psi}_\alpha^\delta(z)]^* \tilde{\Psi}_\alpha^\delta(z) dz,$$

we get $\tilde{\Psi}_\alpha^\delta(z) \underline{e} = 0$ for all z , implying that the columns of $\tilde{\Psi}_\alpha^\delta$ are linearly dependent, which is a contradiction. Hence the proof. \square

Before we state one of our main theorems we introduce some more notation. For $F \in \mathcal{E}^\delta(\mathbb{C}^n)$, define $l(\delta) \times l(\delta)$ matrix $C_\alpha^{\delta, \lambda}(F)$ as follows : Let $\tilde{C}_\alpha^{\delta, \lambda}(F)$ denotes the $\alpha(\delta) \times l(\delta)$ matrix whose r th row is the $l(r)$ th row of $C_\alpha^{\delta, \lambda}$, where $l(r) \in I_1$; and

$\tilde{C}_\alpha^{\delta,\lambda}(F)$ denotes the $(l(\delta) - \alpha(\delta)) \times l(\delta)$ matrix whose r th row is the $m(r)$ th row of $C_\alpha^{\delta,\lambda}$, where $m(r) \in I_2$.

$$\left. \begin{aligned} \tilde{C}_\alpha^{\delta,\lambda}(F) &= (\tilde{A}_\alpha^{\delta,\lambda})^{-1} \int_{\mathbb{C}^n} [\tilde{\Psi}_\alpha^{\delta,\lambda}(z)]^* F(z) dz \\ &= (\tilde{A}_\alpha^{\delta,\lambda})^{-1} \int_{\mathbb{C}^n} [\tilde{L}_\alpha^{\delta,\lambda}(z)]^* \Upsilon_\delta(z) G(z) e^{-|\lambda||z|^2} dz, \\ \tilde{C}_\alpha^{\delta,\lambda}(F) &= 0. \end{aligned} \right\} \quad (7.2)$$

whenever the integrals exist.

Theorem 7.4. (*Hecke-Bochner identity*) Suppose $F = P^\delta G \in \mathcal{S}^\delta(\mathbb{C}^n)$, where G is K -invariant. Then $\mathcal{G}^\lambda(F) = \mathcal{W}^\lambda(P^\delta)S$, where $S \in \mathcal{O}_{l(\delta) \times l(\delta)}(\mathcal{H}^\lambda)$ whose action on each V_α is the $l(\delta) \times l(\delta)$ constant matrix $C_\alpha^{\delta,\lambda}(F)$; equivalently if $F = P^\delta G \in \mathcal{S}^\delta(\mathbb{C}^n)$, where G is K -invariant, then $F \times^\lambda \psi_\alpha^\lambda = \Psi_\alpha^{\delta,\lambda} C_\alpha^{\delta,\lambda}(F)$, where $C_\alpha^{\delta,\lambda}(F)$ is defined by (7.2).

Before proving this general theorem let us consider the special case $K = U(n)$, and describe $\Psi_\alpha^{\delta,\lambda}, \tilde{\Psi}_\alpha^{\delta,\lambda}, L_\alpha^{\delta,\lambda} \dots$. We also show that, in this special case, the above theorem is precisely Theorem 4.2 in [6]. We put these in the following remark.

Remark 7.5. For this remark K always stands for $U(n)$. Let M be the stabilizer of the K -regular point $e_1 = (1, 0, \dots, 0) \in \mathbb{C}^n$; then M can be identified with $U(n-1)$. Via the map $kM \rightarrow k \cdot e_1$, we have the identification $K/M = K \cdot e_1 = S^{2n-1}$. Note that, in this special case, the space H consists of all polynomials P such that $\Delta_{\mathbb{C}^n} P = 0$, where $\Delta_{\mathbb{C}^n} = \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$. for each pair of non-negative integers (p, q) , let \mathcal{P}_{pq} be the space of all polynomials P in z and \bar{z} of the form

$$P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

Let $\mathcal{H}_{pq} = H \cap \mathcal{P}_{pq}$, and \mathcal{S}_{pq} denote the space of restrictions of elements of \mathcal{H}_{pq} to S^{2n-1} . The elements of \mathcal{H}_{pq} are called bigraded solid harmonics of degree (p, q) , and those of \mathcal{S}_{pq} are called bigraded spherical harmonics of degree (p, q) . The K -action on S^{2n-1} defines an unitary representation on $L^2(S^{2n-1})$. Clearly each \mathcal{S}_{pq}

is a K -invariant subspace. Let δ_{pq} denotes the restriction of this representation to \mathcal{S}_{pq} . It is well known that these describe all inequivalent, irreducible, unitary representations in \widehat{K}_M . Note that, according to our general notation (in Section 3), $H_{\delta_{pq}} = \mathcal{H}_{pq}$, $\mathcal{E}_{\delta_{pq}}(K/M) = \mathcal{S}_{pq}$ and $l(\delta_{pq}) = \dim V_{\delta_{pq}}^M = 1$.

Using the similar arguments as in Remark 4.1, we can prove the following : first note that, in this case, δ_{pq}^\vee is equivalent to δ_{qp} . Take $P_{pq}^i \in \mathcal{H}_{pq}$, and Y_{pq}^i to be their restrictions to S^{2n-1} so that $\{Y_{pq}^i : i = 1, 2, \dots, d(p, q)\}$ forms an orthonormal basis for \mathcal{S}_{pq} . Then it is possible to choose orthonormal ordered basis $\mathbf{b} = \{v_1, v_2, \dots, v_{d(p, q)}\}$ for $V_{\delta_{pq}}$ and $\mathbf{b}^M = \{v_1\}$ for $V_{\delta_{pq}}^M$, and a basis \mathbf{e} for $F_{\delta_{pq}} = \text{Hom}(V_{\delta_{pq}}, \mathcal{H}_{pq})$, so that, with respect to these bases, $Y^{\delta_{pq}} : S^{2n-1} \rightarrow \mathcal{M}_{d(p, q) \times 1}$ and $P^{\delta_{pq}} : \mathbb{C}^n \rightarrow \mathcal{M}_{d(p, q) \times 1}$ are given by

$$Y^{\delta_{pq}}(\omega) = \sqrt{\frac{|S^{2n-1}|}{d(p, q)}} \left[Y_{pq}^1(\omega), Y_{pq}^2(\omega), \dots, Y_{pq}^{d(p, q)}(\omega) \right]^t, \omega \in S^{2n-1},$$

$$P^{\delta_{pq}}(z) = \sqrt{\frac{|S^{2n-1}|}{d(p, q)}} \left[P_{pq}^1(z), P_{pq}^2(z), \dots, P_{pq}^{d(p, q)}(z) \right]^t, z \in \mathbb{C}^n.$$

In particular,

$$P^{\delta_{pq}}(z) = |z|^{p+q} Y^{\delta_{pq}}(z/|z|).$$

Also, we have

$$\Upsilon_{\delta_{pq}}^\vee(z) = |z|^{2(p+q)}.$$

(i) Recall the $U(n)$ -spherical functions ψ_k^λ 's from Remark 5.10. The corresponding generalized K -spherical functions are given by $\Psi_k^{\delta_{pq}, \lambda} = \theta^\lambda(P^{\delta_{pq}})\psi_k^\lambda$. Note that, $\Psi_k^{\delta_{pq}, \lambda} : \mathbb{C}^n \rightarrow \mathcal{M}_{d(p, q) \times 1}$, $L_k^{\delta_{pq}, \lambda} : \mathbb{C}^n \rightarrow \mathcal{M}_{1 \times 1}$, and $A_k^{\delta_{pq}, \lambda}$ is a 1×1 matrix. Let L_k^γ denotes the k th degree Laguerre polynomial of type γ . We will show the following

: For $\lambda > 0$,

$$L_k^{\delta_{pq},\lambda}(z) = \begin{cases} (-1)^q \pi^{-n} (2|\lambda|)^{n+p+q} L_{k-p}^{n+p+q-1} (2|\lambda||z|^2), & \text{if } p \leq k \\ 0, & \text{if } p > k, \end{cases} \quad (7.3)$$

$$\Psi_k^{\delta_{pq},\lambda}(z) = \begin{cases} (-1)^q \pi^{-n} (2|\lambda|)^{n+p+q} P^{\delta_{pq}}(z) L_{k-p}^{n+p+q-1} (2|\lambda||z|^2) e^{-|\lambda||z|^2}, & \text{if } p \leq k \\ 0_{d(p,q) \times 1}, & \text{if } p > k, \end{cases} \quad (7.4)$$

$$A_k^{\delta_{pq},\lambda} = \begin{cases} \pi^{-n} (2|\lambda|)^{n+p+q} \frac{\Gamma(k+n+q)}{\Gamma(n)\Gamma(k-p+1)}, & \text{if } p \leq k \\ 0, & \text{if } p > k; \end{cases} \quad (7.5)$$

consequently $\tilde{\Psi}_k^{\delta_{pq},\lambda} = \Psi_k^{\delta_{pq},\lambda}$, $\tilde{L}_k^{\delta_{pq},\lambda} = L_k^{\delta_{pq},\lambda}$ and $\tilde{A}_k^{\delta_{pq},\lambda} = A_k^{\delta_{pq},\lambda}$ if $p \leq k$. When $\lambda < 0$, the role of p and q will be interchanged in the above formulae. We give a proof assuming $\lambda > 0$. The proof for $\lambda < 0$ will be similar. Since

$$\Psi_k^{\delta_{pq},\lambda} = \theta^\lambda (P^{\delta_{pq}}) \psi_k^\lambda = P^{\delta_{pq}}(z) L_k^{\delta_{pq},\lambda}(z) e^{-|\lambda||z|^2},$$

and $z_1^p \bar{z}_2^q \in \mathcal{H}_{pq}$, it follows that

$$\theta^\lambda (z_1^p \bar{z}_2^q) \psi_k^\lambda = z_1^p \bar{z}_2^q L_k^{\delta_{pq},\lambda}(z) e^{-|\lambda||z|^2}.$$

Therefore, to prove (7.3) it is enough to show that

$$\theta^\lambda (z_1^p \bar{z}_2^q) \psi_k^\lambda = \begin{cases} (-1)^q \pi^{-n} (2|\lambda|)^{n+p+q} z_1^p \bar{z}_2^q L_{k-p}^{n+p+q-1} (2|\lambda||z|^2) e^{-|\lambda||z|^2}, & \text{if } p \leq k \\ 0, & \text{if } p > k. \end{cases} \quad (7.6)$$

Since $\theta^\lambda (\bar{z}_2) = \bar{R}_2^\lambda = \partial/\partial z_2 - \lambda \bar{z}_2$, and

$$\frac{\partial}{\partial z_2} [L_k^{n+q-1} (2|\lambda||z|^2)] = 2|\lambda| \bar{z}_2 [L_k^{n+q-1}]' (2|\lambda||z|^2),$$

an easy calculation shows that

$$\theta^\lambda (\bar{z}_2) \psi_k^\lambda(z) = \pi^{-n} (-1) (2|\lambda|)^{n+1} \bar{z}_2 [(L_k^{n-1})' - L_k^{n-1}] (2|\lambda||z|^2) e^{-|\lambda||z|^2}.$$

Using the well-known relations

$$(L_k^\alpha)' = -L_{k-1}^{\alpha+1}, \quad L_k^{\alpha+1} = L_{k-1}^{\alpha+1} + L_k^\alpha,$$

we get

$$\theta^\lambda(\bar{z}_2)\psi_k^\lambda(z) = \pi^{-n}(-1)(2|\lambda|)^{n+1}\bar{z}_2 L_k^{n+1-1}(2|\lambda||z|^2)e^{-|\lambda||z|^2}.$$

Since $\theta^\lambda(\bar{z}_2^{m+1}) = (\bar{R}_2^\lambda)^{m+1} = \bar{R}_2^\lambda \theta^\lambda(\bar{z}_2^m)$, by an induction argument we can prove that

$$\theta^\lambda(\bar{z}_2^q)\psi_k^\lambda(z) = \pi^{-n}(-1)^q(2|\lambda|)^{n+q}\bar{z}_2^q L_k^{n+q-1}(2|\lambda||z|^2)e^{-|\lambda||z|^2},$$

for all non negative integer q . Now fix a q . Then again by induction on p and using a similar argument we can prove that

$$\theta^\lambda(z_1^p \bar{z}_2^q)\psi_k^\lambda(z) = \pi^{-n}(-1)^q(2|\lambda|)^{n+p+q}(z_1^p \bar{z}_2^q) L_{k-p}^{n+p+q-1}(2|\lambda||z|^2)e^{-|\lambda||z|^2},$$

whenever $p \leq k$. In particular we get for any fixed q , $\theta^\lambda(z_1^k \bar{z}_2^q)\psi_k(z)$ is equal to a constant times $z_1^k \bar{z}_2^q e^{-|\lambda||z|^2}$. Since $\theta^\lambda(z_1^{k+1} \bar{z}_2^q) = (-R_1^\lambda)\theta^\lambda(z_1^k \bar{z}_2^q)$ and

$$R_1^\lambda(z_1^k \bar{z}_2^q e^{-|\lambda||z|^2}) = (\partial/\partial \bar{z}_1 + \lambda z_1)(z_1^k \bar{z}_2^q e^{-|\lambda||z|^2}) = 0 \text{ (as } \lambda > 0),$$

we get $\theta(z_1^{k+1} \bar{z}_2^q)\psi_k^\lambda(z) = 0$, and consequently $\theta^\lambda(z_1^p \bar{z}_2^q)\psi_k^\lambda(z) = 0$ for all $p > k$. This finishes the proof of (7.3). (7.4) follows immediately from (7.3). $A_k^{\check{\delta}_{pq}, \lambda}$ has the formulae

$$A_k^{\check{\delta}_{pq}, \lambda} = \int_{\mathbb{C}^n} [L_k^{\check{\delta}_{pq}, \lambda}(z)]^* \Upsilon_{\check{\delta}_{pq}}(z) L_k^{\check{\delta}_{pq}, \lambda}(z) e^{-2|\lambda||z|^2} dz.$$

Therefore, (7.5) follows by (7.3) and the fact that

$$\int_0^\infty [L_k^\gamma(r)]^2 e^{-r} r^\gamma dr = \frac{\Gamma(k + \gamma + 1)}{\Gamma(k + 1)}.$$

(ii) From the above discussion it is immediate that, for the special case $K = U(n)$, Theorem 7.4 can be restated as follows (which is precisely Theorem 4.2 in [6]) : Suppose $Pg \in \mathcal{S}(\mathbb{C}^n)$, where $P \in \mathcal{H}_{pq}$ and g is a radial function. For $\lambda > 0$, $\mathcal{G}^\lambda(Pg) = \mathcal{W}^\lambda(P)S$, $S \in \mathcal{O}(\mathcal{H}^\lambda)$, whose action on each $\mathcal{P}_k(\mathbb{C}^n)$ is a constant c_k , where $c_k = 0$ if $p > k$, and for $p \leq k$, it is given by

$$c_k = (-1)^q \frac{\Gamma(n)\Gamma(k-p+1)}{\Gamma(k+n+q)} \int_{\mathbb{C}^n} g(z) L_{k-p}^{n+p+q-1}(2|\lambda||z|^2) |z|^{2(p+q)} e^{-|\lambda||z|^2}.$$

when $\lambda < 0$, the role of p and q will be interchanged in the definition of c_k .

To prove Theorem 7.4, we need several steps. For a finite dimensional subspace V of \mathcal{H}^λ , let $\mathcal{O}(V)$ stand for the vector space of all bounded linear operators $R : V \longrightarrow \mathcal{H}^\lambda$. Define an inner product \langle, \rangle^λ on $\mathcal{O}(V)$ by

$$\langle R, R' \rangle^\lambda = \sum_{j=1}^d \langle Rv_j, R'v_j \rangle,$$

for an orthonormal basis $\{v_1, v_2, \dots, v_d\}$. Clearly the definition is independent of the orthonormal basis. One can see that the norm defined by the above inner product is equivalent to the operator norm. Since $\mathcal{O}(V)$ is a Banach space with respect to the operator norm, we conclude that $\mathcal{O}(V)$ is a Hilbert space with the above inner product. If $V \subset \mathcal{P}(\mathbb{C}^n)$ we can view $\mathcal{O}(\mathcal{H}^\lambda)$ as a subset of $\mathcal{O}(V)$ by restricting the elements in $\mathcal{O}(\mathcal{H}^\lambda)$ to V . If $V = V_\alpha$, we denote the inner product by $\langle, \rangle_\alpha^\lambda$. In this case, by Schur's orthogonality relation we have another formula for $\langle, \rangle_\alpha^\lambda$, given by

$$\langle R, R' \rangle_\alpha^\lambda = \int_K \langle R(k \cdot v), R'(k \cdot v) \rangle dk ,$$

for any unit vector $v \in V_\alpha$.

Lemma 7.6. (a) Suppose $f \in \mathcal{S}(\mathbb{C}^n)$, and $\delta \in \hat{K}_M$. Then

$$\langle \mathcal{G}^\lambda(f), \mathcal{W}^\lambda(P^\delta) \rangle_\alpha^\lambda = \pi^n (2|\lambda|)^{-n} \langle f, \Psi_\alpha^{\delta, \lambda} \rangle.$$

Here the equality is entry wise.

(b) For two K -harmonic polynomials p, q

$$\langle \mathcal{W}^\lambda(p), \mathcal{W}^\lambda(q) \rangle_\alpha^\lambda = \pi^n (2|\lambda|)^{-n} \langle \theta^\lambda(p) \psi_\alpha^\lambda, \theta^\lambda(q) \psi_\alpha^\lambda \rangle.$$

Proof. The proof is similar to that of Lemma 2.1 in [16].

$$\begin{aligned} \langle \mathcal{G}(f), \mathcal{W}(P^\delta) \rangle_\alpha &= \sum_{\nu=1}^{d(\alpha)} \langle \mathcal{G}(f) e_{\alpha\nu}, \mathcal{W}(P^\delta) e_{\alpha\nu} \rangle \\ &= \sum_{\beta \in \Lambda} \sum_{\nu=1}^{d(\beta)} \langle \mathcal{G}(f) e_{\beta\nu}, \mathcal{W}(P^\delta) \mathcal{P}_\alpha e_{\beta\nu} \rangle \\ &= \sum_{\beta \in \Lambda} \sum_{\nu=1}^{d(\beta)} \langle \mathcal{G}(f) e_{\beta\nu}, \mathcal{G}(\theta(P^\delta)) \mathcal{G}(\psi_\alpha) e_{\beta\nu} \rangle. \end{aligned}$$

By Corollary 5.4,

$$\mathfrak{G}(\theta(P^\delta))\mathfrak{G}(\psi_\alpha) = \mathfrak{G}(\theta(P^\delta)\psi_\alpha) = \mathfrak{G}(\Psi_\alpha^\delta).$$

Hence by the Plancherel Theorem (Theorem 5.1), (a) follows. The proof of (b) is similar. \square

Lemma 7.7. $\{\mathcal{W}^\lambda(p) : p \in \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)\}$ is dense in $\mathcal{O}(V_\alpha)$.

Proof. It is shown in [6] (See Proposition 2.10 (b)) that $\{\mathcal{W}(p) : p \in \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)\}$ is dense in $\mathcal{O}(\mathcal{P}_m(\mathbb{C}^n))$. Since V_α is contained in some $\mathcal{P}_m(\mathbb{C}^n)$, we can extend any operator $T \in \mathcal{O}(V_\alpha)$ to $T' \in \mathcal{O}(\mathcal{P}_m(\mathbb{C}^n))$ by defining T' to be zero on the complement of V_α in $\mathcal{P}_m(\mathbb{C}^n)$. From this, it is easy to see that, $\{\mathcal{W}(p) : p \in \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)\}$ is dense in $\mathcal{O}(V_\alpha)$. \square

Let H_δ^i be the subspace of H_δ , spanned by the entries of i th row in P^δ . Using Schur's orthogonality it can be shown that, if $p \in H_\delta^i$, $q \in H_{\delta'}^{i'}$ with $(\delta, i) \neq (\delta', i')$, then

$$\int_K p(k \cdot z) \overline{q(k \cdot z)} dk = 0. \quad (7.7)$$

Proposition 7.8. $\mathcal{O}(V_\alpha) = \bigoplus_{\delta \in \widehat{K}_M} \bigoplus_{i=1}^{d(\delta)} \mathcal{W}^\lambda(H_\delta^i)|_{V_\alpha}$ (orthogonal Hilbert space decomposition).

Proof. In Lemma 7.6 (b), if we take $p = P_{ij}^\delta$ and $q = P_{i'j'}^{\delta'}$, then $\theta(p)\psi_\alpha = (i, j)$ th entry of $\theta(P^\delta)\psi_\alpha = (i, j)$ th entry of Ψ_α^δ , and similarly $\theta(q)\psi_\alpha = (i', j')$ th entry of $\Psi_\alpha^{\delta'}$. Note that Ψ_α^δ has the form (7.1). Therefore if $(\delta, i) \neq (\delta', i')$, then by (7.7) we have,

$$\int_K [\theta(p)\psi_\alpha](k \cdot z) \overline{[\theta(q)\psi_\alpha](k \cdot z)} dk = 0 \quad \forall z.$$

Integrating both sides over \mathbb{C}^n , and then making a change of variable, namely $z \rightarrow k^{-1} \cdot z$, we get $\langle \theta(p)\psi_\alpha, \theta(q)\psi_\alpha \rangle = 0$ if $(\delta, i) \neq (\delta', i')$. Hence the orthogonality is proved.

By Lemma 7.7, $\{\mathcal{W}(p) : p \in \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)\}$ is dense in $\mathcal{O}(V_\alpha)$. Therefore to complete the proof of the proposition it is enough to show that for any $\delta \in \widehat{K}_M$,

$$\mathcal{W}(IH_\delta)|_{V_\alpha} = \mathcal{W}(H_\delta)|_{V_\alpha}. \quad (7.8)$$

For a polynomial p , let $p(D)$ denote the constant coefficient differential operator obtained by replacing z_j by $-\partial/\partial\bar{z}_j$ and \bar{z}_j by $\partial/\partial z_j$. Then it is an easy consequence of the Euclidean Fourier transform that $\mathcal{F}'^{-1}p = p(D)\delta_0$. Also, we can write

$$\theta(p) = p(D) + \varepsilon(p),$$

where $\varepsilon(p)$ is a polynomial coefficient differential operator of order strictly less than the degree of p . Let P_m be the space of all polynomials in z, \bar{z} whose degree is less than or equal to m . We prove (7.8) by showing that

$$\mathcal{W}(IH_\delta \cap P_m)|_{V_\alpha} \subset \mathcal{W}(H_\delta)|_{V_\alpha}, \quad (7.9)$$

for all non negative integers m . We do it by induction on m . Since $\mathcal{W}(1) = \mathcal{G}(\delta_0) =$ identity operator, (7.9) is true for $m = 0$. Now suppose (7.9) is true for $m = k$. It is enough to show that $\mathcal{W}(p)|_{V_\alpha} \in \mathcal{W}(H_\delta)|_{V_\alpha}$, for any polynomial p of the type $p = jh$, where $j \in I$, $h \in H_\delta$ and $\text{degree } p = (k+1)$.

$$\begin{aligned} \mathcal{W}(p) = \mathcal{G}(\mathcal{F}'^{-1}p) &= \mathcal{G}(h(D)j(D)\delta_0) \\ &= \mathcal{G}(\theta(h)j(D)\delta_0) - \mathcal{G}(\varepsilon(h)j(D)\delta_0) \\ &= \mathcal{W}(h)\mathcal{W}(j) - \mathcal{W}(\mathcal{F}'(\varepsilon(h)j(D)\delta_0)). \end{aligned} \quad (7.10)$$

By Corollary 6.2, $\mathcal{W}(j)$ is a scalar on V_α . Hence

$$[\mathcal{W}(h)\mathcal{W}(j)]|_{V_\alpha} \in \mathcal{W}(H_\delta)|_{V_\alpha}.$$

Now note that $\varepsilon(h)j(D)\delta_0$ is a distribution supported at the origin, whose order is less than or equal to k . Therefore $\mathcal{F}'(\varepsilon(h)j(D)\delta_0)$ is a polynomial of degree at most k . Again from (7.10) we have

$$\mathcal{F}'^{-1}p = \theta(h)j(D)\delta_0 - \varepsilon(h)j(D)\delta_0,$$

which implies that

$$\mathcal{F}'(\varepsilon(h)j(D)\delta_0) \in S_\delta = IH_\delta.$$

Therefore, by the induction hypothesis,

$$\mathcal{W}(\mathcal{F}'(\varepsilon(h)j(D)\delta_0))|_{V_\alpha} \in \mathcal{W}(H_\delta)|_{V_\alpha}.$$

So ultimately we get that

$$\mathcal{W}(p)|_{V_\alpha} \in \mathcal{W}(H_\delta)|_{V_\alpha},$$

as desired. Hence the proof of the proposition is complete. \square

Proof. (Proof of Theorem 7.4) By Lemma 7.6 (a),

$$\langle \mathcal{G}(f), \mathcal{W}(P^{\delta'}) \rangle_\alpha = \pi^n (2|\lambda|)^{-n} \langle f, \Psi_\alpha^{\delta'} \rangle,$$

for $f \in \mathcal{S}(\mathbb{C}^n)$ and $\delta' \in \widehat{K}_M$. Again, $\Psi_\alpha^{\delta'}$ has the form (7.1). Therefore if we take

$$f(z) = F_{ij}(z) = \sum_{k=1}^{l(\delta)} P_{ik}^\delta(z) G_{kj}(z),$$

then by (7.7), we get

$$\langle \mathcal{G}(F_{ij}), \mathcal{W}(P_{i'j'}^{\delta'}) \rangle_\alpha = 0,$$

if $(\delta', i') \neq (\delta, i)$. By Proposition 7.8, in particular for $i = 1$, we get

$$\mathcal{G}(F_{1j})|_{V_\alpha} \in \mathcal{W}(H_{\delta}^1)|_{V_\alpha},$$

for all $j = 1, 2, \dots, l(\delta)$; and consequently there are constants c_{kj} such that

$$\mathcal{G}(F_{1j})|_{V_\alpha} = \sum_{k=1}^{l(\delta)} c_{kj} \mathcal{W}(P_{1k}^\delta)|_{V_\alpha}, \quad (7.11)$$

which implies

$$\begin{aligned} F_{1j} \times \psi_\alpha &= \sum_{k=1}^{l(\delta)} c_{kj} \theta(P_{1k}^\delta) \psi_\alpha \\ &= (1, j)\text{th entry of } [\theta(P^\delta) \psi_\alpha] C_\alpha \\ &= (1, j)\text{th entry of } \Psi_\alpha^\delta C_\alpha, \end{aligned}$$

where C_α is the $l(\delta) \times l(\delta)$ constant matrix whose (k, j) th entry is c_{kj} . Therefore by Corollary 4.4, $F \times \psi_\alpha = \Psi_\alpha^\delta C_\alpha$. This is true for all $\alpha \in \Lambda$. Hence we get $\mathcal{G}(F) = \mathcal{W}(P^\delta)S$, if we define the $l(\delta) \times l(\delta)$ linear operator S by $S|_{V_\alpha} = C_\alpha$.

Since $\Pi(z)^* = \Pi(-z)$, a direct calculation shows that $\mathcal{G}(\bar{f}) = \mathcal{G}(f^-)^*$, for all $f \in \mathcal{S}(\mathbb{C}^n)$, where $f^-(z) = f(-z)$. Also, note that $(L_j f)^- = (-L_j)f^-$ and $(\bar{L}_j f)^- = (-\bar{L}_j)f^-$. Therefore if p, q be two K -harmonic polynomials then

$$\begin{aligned} \mathcal{G}(\theta(p)\psi_\alpha \times \overline{\theta(q)\psi_\beta}) &= \mathcal{G}(\theta(p)\psi_\alpha)\mathcal{G}(\theta(q^-)\psi_\beta^-)^* \\ &= \mathcal{W}(p)\mathcal{G}(\psi_\alpha)[\mathcal{W}(q^-)\mathcal{G}(\psi_\beta^-)]^* \\ &= \tau(p)\mathcal{G}(\psi_\alpha)\mathcal{G}(\psi_\beta)\tau(q^-)^* \\ &= 0, \end{aligned}$$

if $\beta \neq \alpha$. The last two equality holds on the domain of $\tau(q^-)^*$. Since W_j and \bar{W}_j are adjoint to each other we see that $\tau(q^-)^* = \tau(\bar{q}^-)$ whose domain contains $\mathcal{P}(\mathbb{C}^n)$. Hence we get $\theta(p)\psi_\alpha \times \overline{\theta(q)\psi_\beta} = 0$. In particular $\int_{\mathbb{C}^n} \overline{\theta(q)\psi_\beta(z)}\theta(p)\psi_\alpha(z)dz = 0$ if $\beta \neq \alpha$. Applying this, we have for $\beta \neq \alpha$,

$$\int_{\mathbb{C}^n} [\Psi_\beta^\delta(z)]^* \Psi_\alpha^\delta(z) dz = 0. \quad (7.12)$$

Since (by Proposition 6.4)

$$F = \sum_{\beta \in \Lambda} F \times \psi_\beta = \sum_{\beta \in \Lambda} \Psi_\beta^\delta C_\beta,$$

we get

$$\begin{aligned} A_\alpha^\delta C_\alpha &= \int_{\mathbb{C}^n} [\Psi_\alpha^\delta(z)]^* F(z) dz \\ &= \int_{\mathbb{C}^n} [L_\alpha^\delta(z)]^* \Upsilon_\delta(z) G(z) e^{-|\lambda||z|^2} dz. \end{aligned} \quad (7.13)$$

Now we show that it is possible to chose $C_\alpha = C_\alpha^{\delta, \lambda}(F)$, as defined in (7.2). Without loss of generality assume that $l(r) = r$, i.e. first $\alpha(\delta)$ columns in Ψ_α^δ are linearly independent. By Remark 4.5, $\{\Psi_{\alpha 1j}^\delta = \theta(P_{1j}^\delta)\psi_\alpha : j = 1, 2, \dots, \alpha(\delta)\}$ form a basis for $V_{\Psi_\alpha^\delta}^1 = \text{span}\{\Psi_{\alpha 1j}^\delta = \theta(P_{1j}^\delta)\psi_\alpha : j = 1, 2, \dots, l(\delta)\}$; which is equivalent to saying that $\{\mathcal{W}(P_{1j}^\delta)|_{V_\alpha} : j = 1, 2, \dots, \alpha(\delta)\}$ form a basis for $\mathcal{W}(H_\delta^1)|_{V_\alpha}$. Therefore in (7.11) we can take $c_{kj} = 0$ for $k > \alpha(\delta)$. Consequently from (7.13) we get

$$\tilde{A}_\alpha^\delta \tilde{C}_\alpha = \int_{\mathbb{C}^n} [\tilde{L}_\alpha^\delta(z)]^* \Upsilon_\delta(z) G(z) e^{-|\lambda||z|^2} dz,$$

where \tilde{C}_α denotes the $\alpha(\delta) \times l(\delta)$ matrix whose rows are precisely the first $\alpha(\delta)$ rows of C_α . But by Lemma 7.3, $\tilde{A}_\alpha^{\delta,\lambda}$ is invertible. Therefore we can write

$$\tilde{C}_\alpha = (\tilde{A}_\alpha^\delta)^{-1} \int_{\mathbb{C}^n} [\tilde{L}_\alpha^\delta(z)]^* \Upsilon_\delta(z) G(z) e^{-|\lambda||z|^2} dz.$$

Hence by the definition of $C_\alpha^{\delta,\lambda}(F)$, $C_\alpha = C_\alpha^{\delta,\lambda}(F)$ as desired. \square

Now we extend Theorem 7.4 to a larger class of functions. Let

$$\mathcal{E}^\lambda(\mathbb{C}^n) = \{f \in \mathcal{E}(\mathbb{C}^n) : e^{-(|\lambda|-\epsilon)|z|^2} |f(z)| \in L^p(\mathbb{C}^n), \text{ for some } \epsilon > 0, 1 \leq p \leq \infty\},$$

and for $\delta \in \hat{K}_M$,

$$\mathcal{E}^{\delta,\lambda}(\mathbb{C}^n) = \{F \in \mathcal{E}^\delta(\mathbb{C}^n) : \text{each } F_{ij} \in \mathcal{E}^\lambda(\mathbb{C}^n)\}.$$

Since $\psi_\alpha^\lambda(z)$ is equal to $e^{-|\lambda||z|^2}$ times a polynomial, clearly (by Holder's inequality)

$$f \times^\lambda \psi_\alpha^\lambda(z) = \int_{\mathbb{C}^n} f(z-w) \psi_\alpha^\lambda(w) e^{2i\lambda \text{Im}(z \cdot \bar{w})} dw$$

is well defined, whenever $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$. For $\epsilon > 0$ and $z \in \mathbb{C}^n$, define

$$\tau_z^\epsilon \psi_\alpha^\lambda(w) = e^{(|\lambda|-\epsilon)|w|^2} [\psi_\alpha^\lambda(z-w) e^{-2i\lambda \text{Im}(z \cdot \bar{w})}],$$

which clearly belongs to $\mathcal{S}(\mathbb{C}^n)$. Note that if $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$, then for some $\epsilon > 0$, we can think of $e^{-(|\lambda|-\epsilon)|z|^2} f(z)$ as a tempered distribution and then clearly

$$f \times^\lambda \psi_\alpha^\lambda(z) = e^{-(|\lambda|-\epsilon)|z|^2} f(\tau_z^\epsilon \psi_\alpha^\lambda).$$

Lemma 7.9. *Let f be a distribution on \mathbb{C}^n , such that for some $\epsilon > 0$, $e^{-(|\lambda|-\epsilon)|\cdot|^2} f$ is a tempered distribution. Let D be a polynomial coefficient differential operator on \mathbb{C}^n . Then*

- (a) $e^{-(|\lambda|-\epsilon)|\cdot|^2} Df$ is also a tempered distribution.
- (b) Let $f_j \in \mathcal{S}(\mathbb{C}^n)$ be such that $e^{-(|\lambda|-\epsilon)|\cdot|^2} f_j \rightarrow e^{-(|\lambda|-\epsilon)|\cdot|^2} f$ in $\mathcal{S}'(\mathbb{C}^n)$. Then $e^{-(|\lambda|-\epsilon)|\cdot|^2} Df_j \rightarrow e^{-(|\lambda|-\epsilon)|\cdot|^2} Df$ in $\mathcal{S}'(\mathbb{C}^n)$. Consequently for each $z \in \mathbb{C}^n$, $Df_j \times^\lambda \psi_\alpha^\lambda \rightarrow e^{-(|\lambda|-\epsilon)|z|^2} Df(\tau_z^\epsilon \psi_\alpha^\lambda)$.
- (c) In particular, if $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$, $f_j \times^\lambda \psi_\alpha^\lambda(z) \rightarrow f \times^\lambda \psi_\alpha^\lambda(z)$ for each $z \in \mathbb{C}^n$.

Proof. When the action of D on f is multiplication by a polynomial, clearly (a) and (b) are true. Note that

$$e^{-(|\lambda|-\epsilon)|z|^2} \frac{\partial f}{\partial z_j} = \frac{\partial}{\partial z_j} (e^{-(|\lambda|-\epsilon)|z|^2} f) + (|\lambda| - \epsilon) \bar{z}_j f,$$

which immediately proves (a), as well as (b) when $D = \partial/\partial z_j$. General case follows by an induction argument. Assertion (c) is immediate from (b). \square

Theorem 7.10. *Suppose $F = P^\delta G \in \mathcal{E}^{\delta,\lambda}(\mathbb{C}^n)$, G is K -invariant. Then $F \times^\lambda \psi_\alpha^\lambda = \Psi_\alpha^{\delta,\lambda} C_\alpha^{\delta,\lambda}(F)$, where $C_\alpha^{\delta,\lambda}(F)$ is defined by (7.2).*

Proof. Each entry of F belongs to $\mathcal{E}^\lambda(\mathbb{C}^n)$. Take $F_j \in \mathcal{S}^\delta(\mathbb{C}^n)$ such that $e^{-(|\lambda|-\epsilon)|\cdot|^2} F_j \rightarrow e^{-(|\lambda|-\epsilon)|\cdot|^2} F$ entry wise in $\mathcal{S}'(\mathbb{C}^n)$. For each F_j we can apply Theorem 7.4 to get

$$F_j \times \psi_\alpha(z) = \Psi_\alpha^\delta(z) C_\alpha^{\delta,\lambda}(F_j), \quad (7.14)$$

where $C_\alpha^{\delta,\lambda}(F_j)$ is defined by equation (7.2). A similar argument used in the proof of the previous lemma shows that $\lim_{j \rightarrow \infty} C_\alpha^{\delta,\lambda}(F_j) = C_\alpha^{\delta,\lambda}(F)$. On the other hand, by (c) of the previous lemma, for each $z \in \mathbb{C}^n$,

$$\lim_{j \rightarrow \infty} [F_j \times \psi_\alpha(z)] = F \times \psi_\alpha(z).$$

Hence for each $z \in \mathbb{C}^n$, taking limit, as $j \rightarrow \infty$, in (7.14), the proof follows. \square

Now we proceed to prove the uniqueness (upto right multiplication by a constant matrix) of generalized K -spherical function when it belongs to $\mathcal{E}^{\delta,\lambda}(\mathbb{C}^n)$.

Lemma 7.11. *Let $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ be a joint eigenfunction for all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue μ_α^λ . Then $f \times^\lambda \psi_\beta^\lambda = 0$ if $\beta \neq \alpha$.*

Proof. By definition of $\mathcal{E}^\delta(\mathbb{C}^n)$, $e^{-(|\lambda|-\epsilon)|\cdot|^2} f$ is a tempered distribution for some $\epsilon > 0$. Take $f_j \in \mathcal{S}(\mathbb{C}^n)$ such that $e^{-(|\lambda|-\epsilon)|\cdot|^2} f_j \rightarrow e^{-(|\lambda|-\epsilon)|\cdot|^2} f$ in $\mathcal{S}'(\mathbb{C}^n)$. Let $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$. Since $Df = \mu_\alpha(D)f$, by Lemma 7.9 (b),

$$\lim_{j \rightarrow \infty} Df_j \times \psi_\beta(z) = e^{-(|\lambda|-\epsilon)|z|^2} \mu_\alpha(D) f(\tau_z^\epsilon \psi_\beta) = \mu_\alpha(D) f \times \psi_\beta. \quad (7.15)$$

Again by Lemma 7.9 (c),

$$\lim_{j \rightarrow \infty} f_j \times \psi_\beta(z) = f \times \psi_\beta(z), \quad \forall z \in \mathbb{C}^n. \quad (7.16)$$

Now we will show that $Df_j \times \psi_\beta = \mu_\beta(D)f_j \times \psi_\beta$ for all j . By Proposition 5.3 it follows that $\text{range}[\mathcal{G}(D)\mathcal{P}_\beta] \subset \mathbf{P}_N$ for some natural number N . Here \mathbf{P}_N denotes the space of all holomorphic polynomials of degree less than or equal to N . Hence $\mathcal{G}(D)\mathcal{P}_\beta = \left(\sum_{V_\gamma \subset \mathbf{P}_N} \mathcal{P}_\gamma \right) \mathcal{G}(D)\mathcal{P}_\beta$. Enlarging \mathbf{P}_N if necessary we may assume that $V_\beta \subset \mathbf{P}_N$. Therefore we have

$$\mathcal{G}(Df_j \times \psi_\beta) = \mathcal{G}(Df_j)\mathcal{P}_\beta = \mathcal{G}(f_j)\mathcal{G}(D)\mathcal{P}_\beta = \mathcal{G}(f_j) \left(\sum_{V_\gamma \subset \mathbf{P}_N} \mathcal{P}_\gamma \right) \mathcal{G}(D)\mathcal{P}_\beta.$$

But,

$$\mathcal{P}_\gamma \mathcal{G}(D) = \mathcal{G}(\psi_\gamma) \mathcal{G}(D) = \mathcal{G}(D\psi_\gamma) = \mu_\gamma(D) \mathcal{P}_\gamma.$$

Hence

$$\mathcal{G}(Df_j \times \psi_\beta) = \mu_\beta(D) \mathcal{G}(f_j) \mathcal{P}_\beta = \mu_\beta(D) \mathcal{G}(f_j \times \psi_\beta).$$

Therefore $Df_j \times \psi_\beta(z) = \mu_\beta(D)f_j \times \psi_\beta(z)$ for all $z \in \mathbb{C}^n$. Now taking limit as $j \rightarrow \infty$ and using (7.15), (7.16) we get $\mu_\alpha(D)f \times \psi_\beta(z) = \mu_\beta(D)f \times \psi_\beta(z)$. This is true for all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$. Since $\mu_\beta \neq \mu_\alpha$ for $\beta \neq \alpha$, we get $f \times \psi_\beta(z) = 0$ if $\beta \neq \alpha$. Hence the proof. \square

Lemma 7.12. *Let \mathcal{L}^λ be the special Hermite operator and ψ_k^λ 's be the $U(n)$ -spherical functions (see Remark 5.10). Let $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ be an eigenfunction of \mathcal{L}^λ with eigenvalue $-2|\lambda|(2k+n)$. Then $f = f \times^\lambda \psi_k^\lambda$.*

Proof. For this proof let $K = U(n)$ and M be the subgroup of $U(n)$ that fixes the coordinate vector $e_1 = (1, 0, \dots, 0)$ in \mathbb{C}^n . For $\delta \in \widehat{K}_M$, let $\chi_\delta(k) = \text{tr}(\delta(k))$. Define $f_\delta(z) = \int_K f(k^{-1} \cdot z) \chi_\delta(k) dk$. Clearly each f_δ is an eigenfunction of \mathcal{L} with eigenvalue $-2|\lambda|(2k+n)$. Applying the previous lemma for $K = U(n)$ to each f_δ we get $f_\delta \times \psi_m = 0$ if $m \neq k$. Again Proposition 4.5 [16], in particular, implies that each $f_\delta \in \mathcal{S}(\mathbb{C}^n)$. Hence by Proposition 6.4, $f_\delta = \sum_{m \in \mathbb{N}} f_\delta \times \psi_m$. Consequently we get $f_\delta = f_\delta \times \psi_k$ for all $\delta \in \widehat{K}_M$. Since ψ_k is radial, an easy calculation shows

that $(f \times \psi_k)_\delta = f_\delta \times \psi_k$. Therefore $f_\delta = (f \times \psi_k)_\delta$ for all $\delta \in \widehat{K}_M$. But for any smooth function g it is well known that $g(z) = \sum_{\delta \in \widehat{K}_M} g_\delta(z)$, where the right hand side converges uniformly over compact set. Hence we conclude that $f = f \times \psi_k$. \square

Proposition 7.13. *Let $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ be a joint eigenfunction for all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue μ_α^λ . Then $f = f \times^\lambda \psi_\alpha^\lambda$.*

Proof. $V_\alpha \subset \mathcal{P}_k(\mathbb{C}^n)$ for some $k \in \mathbb{N}$. Then f is an eigenfunction of L with eigenvalue $-(2k+n)|\lambda|$. Therefore by Lemma 7.12, $f = f \times \psi_k$. Since $\psi_k = \sum_{V_\beta \subset \mathcal{P}_k(\mathbb{C}^n)} \psi_\beta$, we get $f = \sum_{V_\beta \subset \mathcal{P}_k(\mathbb{C}^n)} f \times \psi_\beta$. But then by Lemma 7.11, $f = f \times \psi_\alpha$. \square

As an immediate consequence of Theorem 7.10 and Proposition 7.13 we get the following Theorem.

Theorem 7.14. *If $\Psi \in \mathcal{E}^{\delta,\lambda}(\mathbb{C}^n)$ is a generalized K -spherical function of type δ corresponding to the eigenvalue μ_α^λ , then $\Psi = \Psi_\alpha^{\delta,\lambda} C$, where $C = C_\alpha^{\delta,\lambda}(\Psi)$ as defined by (7.2).*

We conclude this section by giving another formulae for $\Psi_\alpha^{\delta,\lambda}$ which will be used in the next section. Define

$$\Phi_\alpha^{\delta,\lambda}(z) = \langle \Pi^\lambda(z), \mathcal{W}^\lambda(P^\delta) \rangle_\alpha^\lambda = \sum_{\nu=1}^{d(\alpha)} \langle \Pi^\lambda(z) e_{\alpha\nu}^\lambda, \mathcal{W}^\lambda(P^\delta) e_{\alpha\nu}^\lambda \rangle$$

Proposition 7.15. $\Psi_\alpha^{\delta,\lambda} = \pi^{-n}(2|\lambda|)^n \Phi_\alpha^{\delta,\lambda}$. Consequently $\theta^\lambda(p) \psi_\alpha^\lambda = \langle \Pi^\lambda(z), \mathcal{W}^\lambda(p) \rangle_\alpha^\lambda$ whenever $p \in H_{\delta}^\vee$.

Proof. Note that, on the one hand a direct calculation shows

$$\langle \mathcal{G}^\lambda(f), \mathcal{W}^\lambda(P^\delta) \rangle_\alpha^\lambda = \langle f, \Phi_\alpha^{\delta,\lambda} \rangle,$$

and on the other hand, by Lemma 7.6 (a), we have

$$\langle \mathcal{G}^\lambda(f), \mathcal{W}^\lambda(P^\delta) \rangle_\alpha^\lambda = \pi^n (2|\lambda|)^{-n} \langle f, \Psi_\alpha^{\delta,\lambda} \rangle,$$

for all $f \in \mathcal{S}(\mathbb{C}^n)$. Hence $\Psi_\alpha^{\delta,\lambda} = \pi^{-n}(2|\lambda|)^n \Phi_\alpha^{\delta,\lambda}$. This also can be proved directly using the inversion formulae for Weyl transform. \square

8. K -FINITE EIGENFUNCTIONS

Following the view point of Thangavelu in [16] (see Theorem 3.3 there), we obtain a representation for K -finite joint eigenfunctions in $\mathcal{E}^\lambda(\mathbb{C}^n)$.

Theorem 8.1. *Let $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ be a K -finite joint eigenfunction for all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue μ_α^λ . Then $f(z) = \langle \Pi^\lambda(z), \mathcal{W}^\lambda(P) \rangle_\alpha^\lambda$ for some K -harmonic polynomial P .*

To prove the theorem we first prove the following lemma, which is an easy consequence of Theorem 7.14 and Proposition 7.15.

Lemma 8.2. *Suppose $F : \mathbb{C}^n \longrightarrow \mathcal{M}_{d(\delta) \times d(\delta)}$ is a smooth, square integrable joint eigenfunction for all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue μ_α^λ . Also assume $F(k \cdot z) = \delta(k)F(z)$, for some $\delta \in \widehat{K}_M$. Then, there exists a $l(\delta) \times d(\delta)$ constant matrix C such that $F = \Phi_\alpha^{\delta, \lambda} C$.*

Proof. We suppress the superscript λ . For each $j \in \{1, 2, \dots, d(\delta)\}$, define $F^j : \mathbb{C}^n \longrightarrow \mathcal{M}_{d(\delta) \times l(\delta)}$ to be the matrix whose first column is precisely the j th column of $F(z)$ and else are zero. Then clearly each F^j is square integrable generalized K -spherical function. Hence, by Theorem 7.14 and Proposition 7.15, it follows that there exist $l(\delta) \times l(\delta)$ constant matrix C^j such that $F^j = \Phi_\alpha^\delta C^j$. Equating the entries in first column we get

$$F_{ij} = F_{i1}^j = \sum_{k=1}^{l(\delta)} (\Phi_\alpha^\delta)_{ik} C_{k1}^j, \quad 1 \leq i, j \leq d(\delta);$$

which in matrix form can be written as $F = \Phi_\alpha^\delta C$, where C is the $l(\delta) \times d(\delta)$ constant matrix given by $C_{kj} = C_{k1}^j$. Hence the proof. \square

Let \widehat{K} denote the set of all inequivalent unitary irreducible representations of K . For $\delta \in \widehat{K}$, let $\chi_\delta(k) = \text{tr}[\delta(k)]$.

Proof. (Proof of Theorem 8.1) Since f is K -finite, by Lemma 1.7, Chapter IV of [8], there is a finite subset $\widehat{K}(f)$ of \widehat{K} such that

$$f(z) = \sum_{\delta \in \widehat{K}(f)} d(\delta) \chi_\delta * f(z) := \sum_{\delta \in \widehat{K}(f)} d(\delta) \int_K \chi_\delta(k) f(k^{-1} \cdot z) dk = \sum_{\delta \in \widehat{K}(f)} d(\delta) \text{tr}(f^\delta),$$

where

$$f^\delta(z) = \int_K f(k^{-1} \cdot z) \delta(k) dk.$$

Clearly $f^\delta \in \mathcal{E}^\lambda(\mathbb{C}^n)$. Since any $D \in \mathcal{L}_K(\mathbb{C}^n)$ commutes with the action of K , clearly each f^δ is also a joint eigenfunction for all $D \in \mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue μ_α . Also note that $f^\delta(k \cdot z) = \delta(k) f^\delta(z)$. Now, for $z = (r, kM)$ and $m \in M$,

$$f^\delta(z) = f^\delta(r, kmM) = f^\delta(km \cdot (r, M)) = \delta(k) \delta(m) f^\delta(r, M).$$

Therefore if $\delta \notin \widehat{K}_M$, integrating both side of the above equation over M , we get $f^\delta(z) = 0$. So assume that $\delta \in \widehat{K}_M$. But then by the previous lemma, each f_{ij}^δ can be written as $f_{ij}^\delta(z) = \langle \Pi^\lambda(z), \mathcal{W}^\lambda(\tilde{P}_{ij}^\delta) \rangle_\alpha^\lambda$ for some $\tilde{P}_{ij}^\delta \in H_\delta$. Hence the proof follows. \square

Let $f(z, t)$ be a joint eigenfunction for all $D \in \mathcal{L}_K(\mathfrak{h}_n)$ with eigenvalue $\tilde{\mu}_\alpha^\lambda$. Since $\frac{\partial}{\partial t}(\phi_\alpha^\lambda) = i\lambda\phi_\alpha$, $\tilde{\mu}_\alpha^\lambda(\partial/\partial t) = i\lambda$, f has the form $f(z, t) = e^{i\lambda t}g(z)$. Clearly g is a joint eigenfunction for all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_\alpha^{-\lambda}$. Therefore, Theorem 8.1 implies the following theorem on the Heisenberg group.

Theorem 8.3. *Let f be a K -finite joint eigenfunction for all $D \in \mathcal{L}_K(\mathfrak{h}_n)$ with eigenvalue $\tilde{\mu}_\alpha^\lambda$ such that $f(z, 0) \in \mathcal{E}^\lambda(\mathbb{C}^n)$. Then $f(z, t) = \langle \Pi^\lambda(z, t), \mathcal{W}^\lambda(P) \rangle_\alpha^\lambda$ for some K -harmonic polynomial P .*

The following proposition says that μ_α^λ 's are the only possible eigenvalues for joint eigenfunctions of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$, which belong to $\mathcal{E}^\lambda(\mathbb{C}^n)$. Hence Theorem 8.1, actually describes all K -finite joint eigenfunctions of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$, which belong to $\mathcal{E}^\lambda(\mathbb{C}^n)$. Consequently, Theorem 8.3 actually describes all K -finite joint eigenfunctions $f(z, t)$ of all $D \in \mathcal{L}(\mathfrak{h}_n)$ with eigenvalue $\tilde{\mu}$, such that $\tilde{\mu}(\frac{\partial}{\partial t})$ is a non zero real number and $f(z, 0) \in \mathcal{E}^\lambda(\mathbb{C}^n)$.

Proposition 8.4. *Let $f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ be a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue μ . Then $\mu = \mu_\alpha^\lambda$ for some $\alpha \in \Lambda$.*

Proof. From Remark 7.5, recall \mathcal{S}_{pq} , the space of bigraded spherical harmonics of degree (p, q) . Take an orthonormal basis $\{Y_{pq}^j(\omega) : j = 1, 2, \dots, d(p, q)\}$ for \mathcal{S}_{pq} , so that $\{Y_{pq}^j(\omega) : j = 1, 2, \dots, d(p, q); p + q = k; k = 0, 1, \dots, \infty\}$ form a basis for $L^2(S^{2n-1})$. Therefore for each $r > 0$, $f(r\omega)$ has the bigraded spherical harmonic expansion

$$f(r\omega) = \sum_{m=0}^{\infty} \sum_{p+q=m} \sum_{j=1}^{d(p,q)} f_{pq}^j(r) Y_{pq}^j(\omega), \quad \omega \in S^{2n-1},$$

where

$$f_{pq}^j(r) = \int_{S^{2n-1}} f(r\omega) \overline{Y_{pq}^j(\omega)} d\omega, \quad r > 0.$$

Since f is smooth, clearly $f_{pq}^j(r)$ is bounded at zero. Now let $\mu(\mathcal{L}) = -2|\lambda|(2a + n)$, $a \in \mathbb{C}$. i.e f is an eigenfunction of \mathcal{L} with eigenvalue $-2|\lambda|(2a + n)$. Then it can be shown that (see the proof of Proposition 4.5 [16]), each $f_{pq}^j(r) Y_{pq}^j(\omega)$ is also an eigenfunction of \mathcal{L} with eigenvalue $-2|\lambda|(2a + n)$ i.e

$$\mathcal{L}[f_{pq}^j(r) Y_{pq}^j(\omega)] = -2|\lambda|(2a + n)[f_{pq}^j(r) Y_{pq}^j(\omega)].$$

Writing \mathcal{L} in polar coordinate, using the fact that Y_{pq}^j is an eigenfunction of the spherical Laplacian on S^{2n-1} , and then making a change of variable

$$f_{pq}^j(r) = r^{p+q} u(2|\lambda|r^2) e^{-|\lambda|r^2},$$

we get (for details see the proof of Proposition 4.4 [16]) that u satisfies the following confluent hypergeometric equation

$$tu''(t) + (d - t)u'(t) - (p - a)u(t) = 0, \quad (8.1)$$

where $d = n + p + q$. The equation (8.1) has two linearly independent solutions u_1 and u_2 , with the following asymptotic behaviour (see [1], page-145):

(i) If $(p - a) \neq 0, -1, -2, \dots$,

$$u_1(t) \sim \frac{(d-1)!}{\Gamma(p-a)} e^t t^{p-a-d}, \quad u_2(t) \sim t^{-(p-a)} \text{ as } t \rightarrow +\infty$$

$$u_1(t) \sim 1, \quad u_2(t) \sim \begin{cases} \frac{-\log t}{\Gamma(p-a)} & \text{if } d = 1 \\ \frac{c}{t^{d-1}} & \text{if } d \geq 2 \end{cases} \quad \text{as } t \rightarrow 0^+,$$

where c is a non zero constant.

(ii) If $(p - a)$ is a non positive integer,

$$u_1(t) = L_{a-p}^{d-1}(t), \quad u_2(t) \sim e^t(-t)^{a-p-d} \text{ as } t \rightarrow +\infty.$$

Therefore, under the conditions on f , the only possibility is $(p - a)$ is a non positive integer and consequently

$$f_{pq}^j(r) = r^{p+q} L_{a-p}^{n+p+q-1}(2|\lambda|r^2) e^{-|\lambda|r^2}.$$

So there exists non-positive integer k such that $a = k$. Hence f is a eigenfunction of \mathcal{L} with eigenvalue $-2|\lambda|(2k + n)$. Therefore by Lemma 7.12, $f = f \times \psi_k = \sum_{V_\beta \subset \mathcal{P}_k(\mathbb{C}^n)} f \times \psi_\beta$. Since f is non zero, there exists $\alpha \in \Lambda$ with $V_\alpha \subset \mathcal{P}_k(\mathbb{C}^n)$ such that $f \times \psi_\alpha \neq 0$. Now let $D \in \mathcal{L}_K(\mathbb{C}^n)$. Then

$$Df = \sum_{V_\beta \subset \mathcal{P}_k(\mathbb{C}^n)} D[f \times \psi_\beta] = \sum_{V_\beta \subset \mathcal{P}_k(\mathbb{C}^n)} f \times D\psi_\beta = \sum_{V_\beta \subset \mathcal{P}_k(\mathbb{C}^n)} \mu_\beta(D) f \times \psi_\beta.$$

Again

$$Df = \mu(D)f = \sum_{V_\beta \subset \mathcal{P}_k(\mathbb{C}^n)} \mu(D)f \times \psi_\beta.$$

So we get

$$\sum_{V_\beta \subset \mathcal{P}_k(\mathbb{C}^n)} [\mu(D) - \mu_\beta(D)] f \times \psi_\beta = 0.$$

Hence

$$\sum_{V_\beta \subset \mathcal{P}_k(\mathbb{C}^n)} [\mu(D) - \mu_\beta(D)] f \times \psi_\beta \times \psi_\alpha = 0,$$

which implies that $[\mu(D) - \mu_\alpha(D)] f \times \psi_\alpha = 0$. Since $f \times \psi_\alpha \neq 0$, we get $\mu(D) = \mu_\alpha(D)$. But $D \in \mathcal{L}_K(\mathbb{C}^n)$ is arbitrary. Hence $\mu = \mu_\alpha$. \square

Remark 8.5. Theorem 8.1 holds true even if we assume that f is a distribution such that $e^{-(|\lambda|-\epsilon)|\cdot|^2} f$ defines a tempered distribution for some $\epsilon > 0$.

9. SQUARE INTEGRABLE EIGENFUNCTIONS

In this section we prove the following theorem characterizing square integrable joint eigenfunctions of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$. This is analogous to Theorem 3.3 in [16].

Theorem 9.1. *The square integrable joint eigenfunctions of all the operators $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue μ_α^λ are precisely $f(z) = \langle \Pi^\lambda(z), S \rangle_\alpha^\lambda$, where $S \in \mathcal{O}(V_\alpha)$. Moreover $\|f\|_2^2 = \pi^n (2|\lambda|)^{-n} \|S\|_\alpha^2$.*

Proof. Let $f \in L^2(\mathbb{C}^n)$ be a joint eigenfunction of all $D \in \mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue μ_α . We have

$$f = \sum_{\delta \in \widehat{K}_M} d(\delta) \chi_\delta * f = \sum_{\delta \in \widehat{K}_M} d(\delta) \text{tr}(f^\delta),$$

where the series converges in $L^2(\mathbb{C}^n)$. Clearly each $f^\delta : \mathbb{C}^n \rightarrow \mathcal{M}_{d(\delta) \times d(\delta)}$ is a joint eigenfunction of all $D \in \mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue μ_α . also $f^\delta(k \cdot z) = \delta(k) f^\delta(z)$. Therefore by Lemma 8.2, there is a $(l(\delta) \times d(\delta))$ constant matrix C_δ such that

$$d(\delta) f^\delta = \Psi_\alpha^\delta C_\delta = \pi^{-n} (2|\lambda|)^n \langle \Pi(z), \mathcal{W}(P^\delta) \rangle_\alpha C_\delta.$$

Hence

$$f(z) = \pi^{-n} (2|\lambda|)^n \sum_{\delta \in \widehat{K}_M} \text{tr}[\langle \Pi(z), \mathcal{W}(P^\delta) \rangle_\alpha C_\delta], \quad (9.1)$$

and

$$\begin{aligned} \|f\|_2^2 &= \sum_{\delta \in \widehat{K}_M} \|\text{tr}[\Psi_\alpha^\delta C_\delta]\|_2^2 = \sum_{\delta \in \widehat{K}_M} \|\text{tr}[(\theta(P^\delta) \psi_\alpha) C_\delta]\|_2^2 \\ &= \pi^{-n} (2|\lambda|)^n \sum_{\delta \in \widehat{K}_M} \|\text{tr}[\mathcal{W}(P^\delta) C_\delta]\|_\alpha^2, \end{aligned}$$

where the last equality follows from Lemma 7.6 (b). Therefore

$$S := \pi^{-n} (2|\lambda|)^n \sum_{\delta \in \widehat{K}_M} \text{tr}[\mathcal{W}(P^\delta)|_{V_\alpha} C_\delta]$$

defines an element in $\mathcal{O}(V_\alpha)$, and consequently from (9.1) we get $f(z) = \langle \Pi(z), S \rangle_\alpha$. Conversely let $f(z) = \langle \Pi(z), S \rangle_\alpha$ for some $S \in \mathcal{O}(V_\alpha)$. Let $\widehat{K}(\alpha) = \{\delta \in \widehat{K}_M : \mathcal{W}(H_\delta)|_{V_\alpha} \neq \{0\}\}$. For each $\delta \in \widehat{K}(\alpha)$, choose $p_j^\delta \in H_\delta$, $j = 1, 2, \dots, n_\alpha(\delta)$ so that

$\{\mathcal{W}(p_j^\delta)|_{V_\alpha} : j = 1, 2, \dots, n_\alpha(\delta)\}$ forms an orthonormal basis for $\mathcal{W}(H_\delta)|_{V_\alpha}$. Hence by Proposition 7.8, $\{\mathcal{W}(p_j^\delta)|_{V_\alpha} : j = 1, 2, \dots, n_\alpha(\delta); \delta \in \widehat{K}(\alpha)\}$ is an orthonormal basis for $\mathcal{O}(V_\alpha)$. Therefore we can write for each $z \in \mathbb{C}^n$,

$$\begin{aligned} f(z) = \langle \Pi(z), S \rangle_\alpha &= \sum_{\delta \in \widehat{K}(\alpha)} \sum_{j=1}^{n_\alpha(\delta)} \langle \Pi(z), \mathcal{W}(p_j^\delta) \rangle_\alpha \langle \mathcal{W}(p_j^\delta), S \rangle_\alpha \\ &= \pi^n (2|\lambda|)^{-n} \sum_{\delta \in \widehat{K}(\alpha)} \sum_{j=1}^{n_\alpha(\delta)} [\theta(p_j^\delta) \psi_\alpha](z) \langle \mathcal{W}(p_j^\delta), S \rangle_\alpha \quad (9.2) \end{aligned}$$

by Proposition 7.15. But

$$\sum_{\delta \in \widehat{K}(\alpha)} \sum_{j=1}^{n_\alpha(\delta)} |\langle \mathcal{W}(p_j^\delta), S \rangle_\alpha|^2 = \|S\|_\alpha^2 \leq \infty,$$

and by Lemma 7.6 (b),

$$\langle \theta(p_j^\delta) \psi_\alpha, \theta(p_{j'}^{\delta'}) \psi_\alpha \rangle = \begin{cases} 0 & \text{if } (\delta, j) \neq (\delta', j') \\ \pi^{-n} (2|\lambda|)^n & \text{if } (\delta, j) = (\delta', j'). \end{cases}$$

Therefore it follows that the series for f defined by equation (9.2) converges in $L^2(\mathbb{C}^n)$. In particular $f \in L^2(\mathbb{C}^n)$. Since any $D \in \mathcal{L}_K(\mathbb{C}^n)$ is a polynomial coefficient differential operator we have

$$Df(z) = \pi^n (2|\lambda|)^{-n} \sum_{\delta \in \widehat{K}(\alpha)} \sum_{j=1}^{n_\alpha(\delta)} D[\theta(p_j^\delta) \psi_\alpha](z) \langle \mathcal{W}(p_j^\delta), S \rangle_\alpha$$

in the distribution sense. But $D[\theta(p_j^\delta) \psi_\alpha] = \mu_\alpha(D)[\theta(p_j^\delta) \psi_\alpha]$. Therefore we can conclude that $Df = \mu_\alpha(D)f$. Hence f is a joint eigenfunction of all $D \in \mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue μ_α . Also note that $\|f\|_2^2 = \pi^n (2|\lambda|)^{-n} \|S\|_\alpha^2$. Thus the proof is complete. \square

10. INTEGRAL REPRESENTATIONS OF EIGENFUNCTIONS WHEN $\dim V_\delta^M = 1$

As usual let (K, \mathbb{H}^n) ($K \subset U(n)$) be a Gelfand pair such that the K -action on \mathbb{C}^n is polar. In this section we consider the special case when $\dim V_\delta^M = 1$ for all $\delta \in \widehat{K}_M$, and (under the usual growth condition) characterize any joint eigenfunction

of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$. We show that for this special case it is enough to consider subgroups of the type $K = U(n_1) \times U(n_2) \times \cdots \times U(n_m)$, $n_1 + n_2 + \cdots + n_m = n$. Then the generalized K -spherical functions are given in terms of certain Laguerre polynomials. We use the well-known asymptotic behaviour of Laguerre polynomials to characterize joint eigenfunctions. We give two such characterizations. The first one is a direct generalization of Theorem 8.1 and Theorem 9.1. We will see that this is actually analogous to Theorem 4.1 in [16], which gives an integral representation of eigenfunctions. Though the ideas behind the proof are similar to that in [16], we give the details here, since we will be dealing with $K = U(n_1) \times U(n_2)$ instead of $K = U(n)$. The second one gives a different integral representation of eigenfunctions with an explicit kernel.

Lemma 10.1. *Suppose $\dim V_\delta^M = 1$ for all $\delta \in \widehat{K}_M$. Also assume that the decomposition of $\mathcal{P}_1(\mathbb{C}^n)$ into K -irreducible subspaces is as follows : $\mathcal{P}_1(\mathbb{C}^n) = \bigoplus_{j=1}^m V_j$, where $V_1 = \text{span}\{z_1, z_2, \dots, z_{n_1}\}$, $V_2 = \text{span}\{z_{n_1+1}, z_{n_1+2}, \dots, z_{n_1+n_2}\}$, \dots ; $n_1 + n_2 + \cdots + n_m = n$. Then $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^K = \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^{K_0}$, where $K_0 = U(n_1) \times U(n_2) \cdots \times U(n_m)$.*

Proof. For simplicity of the proof we take $m = 2$. Let $\{v_1, v_2, \dots, v_{d(\alpha)}\}$ be an orthonormal (in \mathcal{H}^λ for $\lambda = \frac{1}{2}$) basis for V_α and $p_\alpha = \sum_{i=1}^{d(\alpha)} v_i \bar{v}_i$. Then $\{p_\alpha\}_{\alpha \in \Lambda}$ is a vector space basis for $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^K$ (see Proposition 3.9 in [2]). In particular, any K -invariant second degree homogeneous polynomial has to be a linear combination of γ_1 and γ_2 , where $\gamma_1(z) = |z_1|^2 + |z_2|^2 + \cdots + |z_{n_1}|^2$ and $\gamma_2(z) = |z_{n_1+1}|^2 + |z_{n_1+2}|^2 + \cdots + |z_n|^2$. Since $U(n_1) \times U(n_2)$ invariant elements in $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)$ are generated by γ_1 and $\gamma_2(z)$, to prove the theorem it is enough to show that any K -invariant homogeneous polynomial can be written as a polynomial in $\gamma_1(z)$ and $\gamma_2(z)$. We prove this by induction on the degree of K -invariant homogeneous polynomials. Note that degree of a K -invariant homogeneous polynomial is always even. Denote the representation of K on V_j by δ_j . Since $l(\delta_j) = \dim V_j^M = 1$, $H_{\delta_j} = V_j$ (see (3.2)). Now if $V_\alpha \subset \mathcal{P}_2(\mathbb{C}^n)$ then degree of p_α is 4. Since $\mathcal{F}'\left(\frac{\partial p_\alpha}{\partial \bar{z}_1}\right)$ is equal to z_1 times a K -invariant distribution, which transform according to representation δ_1 and K -action commutes with \mathcal{F}' ,

the same is true for $\frac{\partial p_\alpha}{\partial \bar{z}_1}$. Hence $\frac{\partial p_\alpha}{\partial \bar{z}_1} \in IH_{\delta_1} = IV_1$, where $I = \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^K$. But $\frac{\partial p_\alpha}{\partial \bar{z}_1}$ being a third degree homogeneous polynomial and any K -invariant second degree homogeneous polynomial being a linear combination of γ_1 and γ_2 , $\frac{\partial p_\alpha}{\partial \bar{z}_1}$ has to be of the following form :

$$\frac{\partial p_\alpha}{\partial \bar{z}_1}(z) = \sum_{j=1}^{n_1} z_j [a_j \gamma_1(z) + b_j \gamma_2(z)]. \quad (10.1)$$

Similarly, as $\frac{\partial p_\alpha}{\partial \bar{z}_{n_1+1}}$ is a third degree homogeneous polynomial which belongs to $IH_{\delta_2} = IV_2$, it has the following form :

$$\frac{\partial p_\alpha}{\partial \bar{z}_{n_1+1}}(z) = \sum_{j=1}^{n_2} z_{n_1+j} [a'_j \gamma_1(z) + b'_j \gamma_2(z)]. \quad (10.2)$$

From the above two equations we get

$$\frac{\partial^2 p_\alpha}{\partial \bar{z}_{n_1+1} \partial \bar{z}_1} = \sum_{j=1}^{n_1} b_j z_j z_{n_1+1} = \sum_{j=1}^{n_2} a'_j z_{n_1+j} z_1,$$

which implies that $b_j = 0$ if $j \neq 1$. So (10.1) becomes

$$\frac{\partial p_\alpha}{\partial \bar{z}_1}(z) = z_1 [a_1 \gamma_1(z) + b_1 \gamma_2(z)] + \sum_{j=2}^{n_1} z_j a_j \gamma_1(z). \quad (10.3)$$

Now let

$$v_i(z) = z_1 (c_{i1} z_1 + c_{i2} z_2 + \cdots c_{in} z_n) + q_1(z), \quad i = 1, 2, \dots, d(\alpha),$$

where $q_1(z)$ is a second degree homogeneous holomorphic polynomial in z_2, z_3, \dots, z_n .

Then

$$\frac{\partial p_\alpha}{\partial \bar{z}_1}(z) = \sum_{i=1}^{d(\alpha)} [z_1 (c_{i1} z_1 + c_{i2} z_2 + \cdots c_{in} z_n) + q_1(z)] [2\bar{c}_{i1} \bar{z}_1 + \bar{c}_{i2} \bar{z}_2 + \cdots \bar{c}_{in} \bar{z}_n]. \quad (10.4)$$

Equating the coefficient of $z_2 z_1 \bar{z}_1$ from the right hand sides of (10.3) and (10.4) we get $a_2 = 2 \sum c_{i2} \bar{c}_{i1}$. Again equating the coefficients of $z_1^2 \bar{z}_2$ from the right hand sides of (10.3) and (10.4) we get $\sum c_{i1} \bar{c}_{i2} = 0$. Hence $a_2 = 0$. similarly $a_3 = a_4 = \cdots = a_{n_1} = 0$.) So from (10.3) we get

$$\frac{\partial p_\alpha}{\partial \bar{z}_1}(z) = z_1 [a_1 \gamma_1(z) + b_1 \gamma_2(z)].$$

Similarly we get

$$\frac{\partial p_\alpha}{\partial \bar{z}_j}(z) = z_j[a_{j1}\gamma_1(z) + b_{j1}\gamma_2(z)], j = 1, 2, \dots, n.$$

Hence we can write p_α as

$$p_\alpha(z) = \begin{cases} a_j\gamma_1(z)^2 + b_j\gamma_1(z)\gamma_2(z) + r_j(z) & \text{if } j = 1, 2, \dots, n_1 \\ c_j\gamma_1(z)\gamma_2(z) + d_j\gamma_2(z)^2 + r_j(z) & \text{if } j = n_1 + 1, n_1 + 2, \dots, n_1 + n_2 = n. \end{cases}$$

Here $r_j(z)$ is a fourth degree homogeneous polynomial in z, \bar{z} , which is independent of \bar{z}_j . For $j = 1, 2$, equating the coefficients of $|z_1|^2|z_2|^2$, we get $a_1 = a_2$. Similarly we can show that all a_j 's are same; and all d_j 's are same. Again for $j = i$ and $j = n_1 + k$ ($i = 1, 2, \dots, n_1; k = 1, 2, \dots, n_2$), equating the coefficients of $|z_i|^2|z_{n_1+k}|^2$, we get $b_i = c_{n_1+k}$. Hence we can write

$$p_\alpha(z) = a_1\gamma_1(z)^2 + b_1\gamma_1(z)\gamma_2(z) + d_1\gamma_2(z)^2 + \tilde{r}_j(z), \forall j = 1, 2, \dots, n,$$

where $\tilde{r}_j(z)$ is a fourth degree homogeneous polynomial in z, \bar{z} , which is independent of \bar{z}_j . Therefore

$$p_\alpha(z) = a_1\gamma_1(z)^2 + b_1\gamma_1(z)\gamma_2(z) + d_1\gamma_2(z)^2 + r(z),$$

where $r(z) (= \tilde{r}_1(z) = \tilde{r}_2(z) = \dots = \tilde{r}_n(z))$ is a fourth degree homogeneous polynomial in z only. But since p_α has the form $p_\alpha = \sum_{i=1}^{d(\alpha)} v_i \bar{v}_i$, it follows that $r(z) \equiv 0$. Hence

$$p_\alpha(z) = a_1\gamma_1(z)^2 + b_1\gamma_1(z)\gamma_2(z) + d_1\gamma_2(z)^2.$$

So we have proved that if $V_\alpha \subset \mathcal{P}_2(\mathbb{C}^n)$, then p_α can be written as a polynomial in γ_1 and γ_2 . Hence, it follows that any K -invariant, 4th degree, homogeneous polynomial can be written as a polynomial in γ_1 and γ_2 . Now, let any K -invariant homogeneous polynomial of degree $2N$ can be written as a polynomial in γ_1 and γ_2 . We have to show that any K -invariant homogeneous polynomial of degree $2(N+1)$ can be written as a polynomial in γ_1 and γ_2 . But for this, it is enough to show the following : If $V_\alpha \subset \mathcal{P}_{N+1}(\mathbb{C}^n)$, then p_α can be written as a polynomial in γ_1 and γ_2 . So fix a $\alpha \in \Lambda$, such that $V_\alpha \subset \mathcal{P}_{N+1}(\mathbb{C}^n)$. Then $\frac{\partial p_\alpha}{\partial \bar{z}_1}$ is a $(2N+1)$ th

degree homogeneous polynomial which belongs to IV_1 . Therefore by the induction hypothesis, $\frac{\partial p_\alpha}{\partial \bar{z}_1}(z)$ has the following form

$$\frac{\partial p_\alpha}{\partial \bar{z}_1}(z) = \sum_{j=1}^{n_1} z_j \left[\sum_{l=0}^N a_j^l (\gamma_1(z))^{N-l} (\gamma_2(z))^l \right].$$

Similarly, $\frac{\partial p_\alpha}{\partial \bar{z}_{n_1+1}}$ has the following form

$$\frac{\partial p_\alpha}{\partial \bar{z}_{n_1+1}}(z) = \sum_{j=1}^{n_2} z_{n_1+j} \left[\sum_{l=0}^N b_j^l (\gamma_1(z))^{N-l} (\gamma_2(z))^l \right].$$

Now using similar arguments as before, it is possible to show that p_α is generated by γ_1 and γ_2 . Hence the proof. \square

Proposition 10.2. *Suppose $\dim V_\delta^M = 1$ for all $\delta \in \widehat{K}_M$. Then there exist $J \in U(n)$, and positive integers n_1, n_2, \dots, n_m with $n_1 + n_2 + \dots + n_m = n$, such that $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^K = \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^{K_0}$, where $K_0 = J[U(n_1) \times U(n_2) \times \dots \times U(n_m)]J^{-1}$.*

Proof. Let $\mathcal{P}_1(\mathbb{C}^n) = \bigoplus_{l=1}^m V_l$ be the decomposition of $\mathcal{P}_1(\mathbb{C}^n)$ into K -irreducible subspaces. V_l 's are pairwise orthogonal in $\mathcal{H}^{\frac{1}{2}}$. Let $\dim V_l = n_l$ so that $n_1 + n_2 + \dots + n_m = n$. Let $u_i(z) = z_i$. Choose an orthonormal (in $\mathcal{H}^{\frac{1}{2}}$) basis $\{v_i(z) = \sum_{j=1}^n c_{ij} u_j : i = 1, 2, \dots, n\}$ for $\mathcal{P}_1(\mathbb{C}^n)$ such that $\{v_1, v_2, \dots, v_{n_1}\}$, $\{v_{n_1+1}, v_{n_1+2}, \dots, v_{n_1+n_2}\}$, \dots form a basis for V_1, V_2, \dots respectively. Since $\{u_1(z), u_2(z), \dots, u_n(z)\}$ is an orthonormal set in $\mathcal{H}^{\frac{1}{2}}$, we get $\sum_{j=1}^n |c_{ij}|^2 = 1$; and $\sum_{j=1}^n c_{ij} \overline{c_{i'j}} = 0$ if $i \neq i'$. Hence the matrix $J := (\bar{c}_{ij})_{n \times n}$ is unitary. Therefore $J^{-1} = J^* = (c_{ji})_{n \times n}$, which implies that $(J \cdot u_i)(z) = u_i(J^{-1} \cdot z) = v_i(z)$ or $J \cdot u_i = v_i$. Consequently the decomposition of $\mathcal{P}_1(\mathbb{C}^n)$ into $J^{-1}KJ$ -irreducible subspaces is given by $\mathcal{P}_1(\mathbb{C}^n) = \bigoplus_{l=1}^m V'_l$, where $V'_1 = \text{span}\{u_1, u_2, \dots, u_{n_1}\}$, $V'_2 = \text{span}\{u_{n_1+1}, u_{n_1+2}, \dots, u_{n_1+n_2}\} \dots$. Next, $M = K_{z_0}$ for some K -regular point z_0 . Then clearly $J^{-1} \cdot z_0$ is a $J^{-1}KJ$ -regular point, and $J^{-1}MJ = [J^{-1}KJ]_{J^{-1} \cdot z_0}$. Let $K' = J^{-1}KJ$ and $M' = J^{-1}MJ$. For each $\delta \in \widehat{K}_M$ define the irreducible unitary representation δ' of K' on $V_{\delta'} = V^\delta$ by $\delta'(J^{-1}kJ) = \delta(k)$ for all $k \in K$. Then it is easy to see that the map $\delta \rightarrow \delta'$ is a bijection from \widehat{K}_M onto $\widehat{K}'_{M'}$, and $\dim V_{\delta'}^{M'} = 1$ for all $\delta' \in \widehat{K}'_{M'}$. Therefore by

the previous lemma $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^{K'} = \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^{K'_0}$, where $K'_0 = U(n_1) \times U(n_2) \cdots \times U(n_m)$. Hence the proof. \square

If $\dim V_{\delta}^M = 1$ for all $\delta \in \widehat{K}_M$, the above proposition says that with respect to a suitable coordinate system on \mathbb{C}^n , the K -invariant polynomials are same as that of $U(n_1) \times U(n_2) \times \cdots \times U(n_m)$, $n_1 + n_2 + \cdots + n_m = n$. Since $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^K$ determines $\mathcal{L}_K(\mathbf{h}_n)$ (see [2], section-3), and hence $\mathcal{L}_K^{\lambda}(\mathbb{C}^n)$, to find joint eigenfunctions of all $D \in \mathcal{L}_K^{\lambda}(\mathbb{C}^n)$ for this special case, it is enough to consider the groups $U(n_1) \times U(n_2) \times \cdots \times U(n_m)$, $n_1 + n_2 + \cdots + n_m = n$. For simplicity of notation, here we only deal with the particular case : $m = 2$.

So, from now on K always stands for $U(n_1) \times U(n_2)$, and M the stabilizer of the K -regular point $e = (1, 0, \dots, 1, 0, \dots, 0) \in \mathbb{C}^n$, where the second 1 is at the $(n_1 + 1)$ th position. Via the map $kM \rightarrow k \cdot e$, we have the identification

$$K/M = K \cdot e = \{z \in \mathbb{C}^n : \sum_{j=1}^{n_1} |z_j|^2 = 1, \sum_{j=n_1+1}^n |z_j|^2 = 1\}.$$

If we identify \mathbb{C}^n with $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ by the map $z \rightarrow ((z_1, z_2, \dots, z_{n_1}), (z_{n_1+1}, z_{n_1+2}, \dots, z_n))$, then $K/M = S^{2n_1-1} \times S^{2n_2-1}$, where S^{2n_1-1} is the unit sphere in \mathbb{C}^{n_1} and S^{2n_2-1} is the unit sphere in \mathbb{C}^{n_2} . Now we explicitly describe the spaces H_{δ} and $\mathcal{E}_{\delta}(K/M) = \mathcal{E}_{\delta}(S^{2n_1-1} \times S^{2n_2-1})$. Since I_+ , the set of polynomials in $I = \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n)^K$ without constant term, is generated by $\sum_{j=1}^{n_1} |z_j|^2, \sum_{j=n_1+1}^n |z_j|^2$, the set of K -harmonic polynomials is given by

$$H = \{P \in \mathcal{P}(\mathbb{C}_{\mathbb{R}}^n) : \Delta_1 P = 0, \Delta_2 P = 0\},$$

where

$$\Delta_1 = \sum_{j=1}^{n_1} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}, \quad \Delta_2 = \sum_{j=n_1+1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j}.$$

For $z \in \mathbb{C}^n$, let $z^1 = (z_1, z_2, \dots, z_{n_1}) \in \mathbb{C}^{n_1}$, $z^2 = (z_{n_1+1}, z_{n_1+2}, \dots, z_n)$. Let $i = 1$ or 2 . For each pair of positive integer (p, q) , we define \mathcal{P}_{pq}^i to be the subspace of $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^{n_i})$ consisting of all polynomials of the form

$$P(z^i) = \sum_{|\alpha_i|=p} \sum_{|\beta_i|=q} (z^i)^{\alpha_i} (\bar{z}^i)^{\beta_i}.$$

Here α_i and β_i are multi-indices of non negative integers of length n_i . We let

$$H^i = \{P \in \mathcal{P}(\mathbb{C}_{\mathbb{R}}^{n_i}) : \Delta_i P = 0\}; \quad H_{pq}^i = \{P \in \mathcal{P}_{pq}^i : \Delta_i P = 0\}.$$

We have the identification $\mathcal{P}(\mathbb{C}_{\mathbb{R}}^n) = \mathcal{P}(\mathbb{C}_{\mathbb{R}}^{n_1}) \otimes \mathcal{P}(\mathbb{C}_{\mathbb{R}}^{n_2})$, $H = H^1 \otimes H^2$, and consequently H has the algebraic direct sum decomposition :

$$H = \bigoplus_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+} H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2.$$

Here \mathbb{Z}^+ denotes the set of non negative integers. Also note that each $P \in \mathcal{P}_{p_1 q_1}^1 \otimes \mathcal{P}_{p_2 q_2}^2$ satisfy the homogeneity condition

$$P(\lambda_1 z^1, \lambda_2 z^2) = \lambda_1^{p_1} \bar{\lambda}_1^{q_1} \lambda_2^{p_2} \bar{\lambda}_2^{q_2} P(z)$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$. Let \mathcal{E}_{pq}^i stand for the restrictions of members of H_{pq}^i to S^{2n_i-1} . The relation between $P \in H_{pq}^i$ and its restriction Y_{pq}^i is given by $P_{pq}^i(z^i) = |z^i|^{p+q} Y_{pq}^i(\omega^i)$, if $z^i = r^i \omega^i$, $r^i > 0$, $\omega^i \in S^{2n_i-1}$. The natural action of $U(n_i)$ defines a unitary representation, δ_{pq}^i on each of these spaces \mathcal{E}_{pq}^i , considered as a Hilbert subspace of $L^2(S^{2n_i-1})$. Clearly the restriction of $H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2$ to $S^{2n_1-1} \times S^{2n_2-1}$ is given by $\mathcal{E}_{p_1 q_1}^1 \otimes \mathcal{E}_{p_2 q_2}^2$. If we consider this as a Hilbert subspace of $L^2(S^{2n_1-1} \times S^{2n_2-1})$, then the natural action of K on each of these spaces $\mathcal{E}_{p_1 q_1}^1 \otimes \mathcal{E}_{p_2 q_2}^2$ defines a unitary representation which is same as $\delta_{p_1 q_1}^1 \otimes \delta_{p_2 q_2}^2$. Now for each fixed $i \in \{1, 2\}$, we have the following well known facts about the class one representations of $U(n_i)$ (see [17], page: 64-69) : The representations δ_{pq}^i of $U(n_i)$ on \mathcal{E}_{pq}^i are irreducible. δ_{pq}^i and $\delta_{p'q'}^i$ are unitarily equivalent if and only if $(p, q) = (p', q')$. Let $M_i \subset U(n_i)$ be the stabilizer of $(1, 0, \dots, 0) \in \mathbb{C}^{n_i}$, so that $M = M_1 \times M_2$. Any $\delta \in \widehat{U(n_i)}_{M_i}$ is equivalent to some δ_{pq}^i . $L^2(S^{2n_i-1})$ has the orthogonal Hilbert space decomposition : $L^2(S^{2n_i-1}) = \bigoplus_{p, q \in \mathbb{Z}^+}^\perp \mathcal{E}_{pq}^i$. From these facts we can prove the following proposition.

Proposition 10.3. (a) *The representations $\delta_{p_1 q_1}^1 \otimes \delta_{p_2 q_2}^2$ of K on $\mathcal{E}_{p_1 q_1}^1 \otimes \mathcal{E}_{p_2 q_2}^2$ are irreducible. $\delta_{p_1 q_1}^1 \otimes \delta_{p_2 q_2}^2$ and $\delta_{p'_1 q'_1}^1 \otimes \delta_{p'_2 q'_2}^2$ are unitarily equivalent if and only if $(p_1, q_1, p_2, q_2) = (p'_1, q'_1, p'_2, q'_2)$. Moreover any $\delta \in \hat{K}_M$ is equivalent to some $\delta_{p_1 q_1}^1 \otimes \delta_{p_2 q_2}^2$.*

(b) We have the orthogonal Hilbert space decomposition of $L^2(S^{2n_1-1} \times S^{2n_2-1})$:

$$L^2(S^{2n_1-1} \times S^{2n_2-1}) = \bigoplus_{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+}^{\perp} \mathcal{E}_{p_1 q_1}^1 \otimes \mathcal{E}_{p_2 q_2}^2.$$

By the above proposition, the decomposition of $\mathcal{P}(\mathbb{C}^n)$ into K -irreducible subspaces is given by

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{m_1, m_2 \in \mathbb{Z}^+} V_{m_1 m_2},$$

where

$$V_{m_1 m_2} = \text{span}\{(z^1)^{\alpha_1} (z^2)^{\alpha_2} : |\alpha_1| = m_1, |\alpha_2| = m_2\}.$$

Denote the corresponding bounded K -spherical functions by $\phi_{m_1 m_2}^\lambda(z, t) = e^{i\lambda t} \psi_{m_1 m_2}^\lambda(z)$.

Then $\psi_{m_1 m_2}^\lambda(z)$ is a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue, say $\mu_{m_1 m_2}^\lambda$.

Note that here $\mathcal{L}_K^\lambda(\mathbb{C}^n)$ is generated by

$$\mathcal{L}_1^\lambda := \sum_{j=1}^{n_1} L_j^\lambda \bar{L}_j^\lambda + \bar{L}_j^\lambda L_j^\lambda, \text{ and } \mathcal{L}_2^\lambda := \sum_{j=n_1+1}^{n_2} L_j^\lambda \bar{L}_j^\lambda + \bar{L}_j^\lambda L_j^\lambda.$$

Let L_k^α be the k th degree Laguerre polynomial of type α . For any $\nu \in \mathbb{N}$, and any $\zeta \in \mathbb{C}^\nu$, define

$$\varphi_{k, \lambda}^\alpha(\zeta) = L_k^\alpha(2|\lambda||\zeta|^2) e^{-|\lambda||\zeta|^2}.$$

Proposition 10.4. $\mu_{m_1 m_2}^\lambda(\mathcal{L}_i^\lambda) = -2|\lambda|(2m_i + n_i)$, $i = 1, 2$. $\psi_{m_1 m_2}^\lambda$ has the following formulae in terms of Laguerre polynomials :

$$\psi_{m_1 m_2}^\lambda(z) = \pi^{-n} (2|\lambda|)^n \prod_{i=1}^2 \varphi_{m_i, \lambda}^{n_i-1}(z^i).$$

Proof. As usual we drop the superscript λ . Take $z_1^{m_1} z_{n_1+1}^{m_2} \in V_{m_1 m_2}$. By Remark 5.9,

$$\mathcal{G}(\mathcal{L}_1)[z_1^{m_1} z_{n_1+1}^{m_2}] = \mu_{m_1 m_2}(\mathcal{L}_1)[z_1^{m_1} z_{n_1+1}^{m_2}],$$

which, by Proposition 5.3, reduces to

$$\left(- \sum_{j=1}^{n_1} \bar{W}_j W_j + W_j \bar{W}_j \right) [z_1^{m_1} z_{n_1+1}^{m_2}] = \mu_{m_1 m_2}(\mathcal{L}_1) [z_1^{m_1} z_{n_1+1}^{m_2}].$$

Using the definition of W_j and \bar{W}_j , an easy calculation shows that $\mu_{m_1 m_2}(\mathcal{L}_1) = -2|\lambda|(2m_1 + n_1)$. Similarly $\mu_{m_1 m_2}(\mathcal{L}_2) = -2|\lambda|(2m_2 + n_2)$. Since $\varphi_{m_i, \lambda}^{n_i-1}(z^i)$ is an

eigenfunction of \mathcal{L}_i with eigenvalue $-2|\lambda|(2m_i + n_i)$, $\prod_{i=1}^2 \varphi_{m_i, \lambda}^{n_i-1}(z^i)$ is a joint eigenfunction of $\mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}$. Hence $\psi_{m_1 m_2}(z) = c \prod_{i=1}^2 \varphi_{m_i, \lambda}^{n_i-1}(z^i)$, for some constant c . To calculate the constant c , first note that by Proposition 6.4, $\psi_{m_1 m_2}(z) = \psi_{m_1 m_2} \times \psi_{m_1 m_2}(z)$. In particular, putting $z = 0$, we get

$$c L_{m_1}^{n_1-1}(0) L_{m_2}^{n_2-1}(0) = c^2 \prod_{i=1}^2 \int_{\mathbb{C}^{n_i}} [L_{m_i}^{n_i-1}(2|\lambda||z^i|^2)]^2 e^{-2|\lambda||z^i|^2} dz^i.$$

Using the well-known facts

$$L_k^\alpha(0) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha + 1)}, \text{ and } \int_0^\infty [L_k^\alpha(r)]^2 e^{-r} r^\alpha dr = \frac{\Gamma(k + \alpha + 1)}{\Gamma(k + 1)},$$

we can deduce that $c = \pi^{-n}(2|\lambda|)^n$. Hence the proof. \square

From now on we always assume that $\lambda > 0$ and state our results only for $\lambda > 0$. The corresponding results for $\lambda < 0$ can be obtained by interchanging the role of p_i and q_i .

Proposition 10.5. *Let $P \in H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2$. Then*

$$\theta^\lambda(P) \psi_{m_1 m_2}^\lambda(z) = \pi^{-n} (2|\lambda|)^n P(z) \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i + q_i} \varphi_{m_i - p_i, \lambda}^{n_i + p_i + q_i - 1}(z^i)$$

if $p_i \leq m_i$ for all $i = 1, 2$; otherwise $\theta^\lambda(P) \psi_{m_1 m_2}^\lambda(z) = 0$.

Proof. Since $\delta_{p_1 q_1}^1 \otimes \delta_{p_2 q_2}^2$ has a unique (upto a constant multiple) M -fixed vector, by Corollary 4.4, it is enough to prove the proposition for $P(z) = z_1^{p_1} \bar{z}_2^{q_1} z_{n_1+1}^{p_2} \bar{z}_{n_1+2}^{q_2}$, which clearly belongs to $H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2$. Since $\theta(P) = \bar{R}_2^{q_1} \bar{R}_{n_1+2}^{q_2} (-R_1)^{p_1} (R_{n_1+1})^{q_2}$, it follows that

$$\theta(P) \psi_{m_1 m_2} = [\theta(z_1^{p_1} \bar{z}_2^{q_1}) \psi_{m_1}(z^1)] [\theta(z_{n_1+1}^{p_2} \bar{z}_{n_1+2}^{q_2}) \psi_{m_2}(z^2)],$$

where

$$\psi_{m_i}(z^i) = \pi^{-n_i} (2|\lambda|)^{n_i} \varphi_{m_i, \lambda}^{n_i-1}(z^i), \quad i = 1, 2.$$

Hence the proof follows by (7.6). \square

Corollary 10.6. (a) Let $Pg \in \mathcal{E}^\lambda(\mathbb{C}^n)$ be a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$, where g is K -invariant and $P \in H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2$. Then there is a constant $c_{p_1 q_1 p_2 q_2}$ such that

$$P(z)g(z) = c_{p_1 q_1 p_2 q_2} P(z) \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i+q_i} \varphi_{m_i-p_i, \lambda}^{n_i+p_i+q_i-1}(z^i),$$

when $p_i \leq m_i$ for all $i = 1, 2$; otherwise $Pg = 0$.

(b) Let $P \in H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2$. Then

$$\langle \Pi^\lambda(z), \mathcal{W}^\lambda(P) \rangle_{m_1 m_2}^\lambda = P(z) \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i+q_i} \varphi_{m_i-p_i, \lambda}^{n_i+p_i+q_i-1}(z^i),$$

when $p_i \leq m_i$ for all $i = 1, 2$; otherwise $\langle \Pi^\lambda(z), \mathcal{W}^\lambda(P) \rangle_{m_1 m_2}^\lambda = 0$.

Proof. Since $\delta_{p_1 q_1}^1 \otimes \delta_{p_2 q_2}^2$ has unique (upto a constant multiple) M -fixed vector, (a) follows from Theorem 7.14 and Proposition 10.5; (b) follows from Proposition 7.15 and Proposition 10.5. \square

Lemma 10.7. Let $\{Y_{p_1 q_1 p_2 q_2}^j : j = 1, 2, \dots, d(p_1, q_1, p_2, q_2)\}$ be an orthonormal basis for $\mathcal{E}_{p_1 q_1}^1 \otimes \mathcal{E}_{p_2 q_2}^2$ so that $\{Y_{p_1 q_1 p_2 q_2}^j : j_i = 1, 2, \dots, d(p_1, q_1, p_2, q_2); p_1, q_1, p_2, q_2 \in \mathbb{Z}^+\}$ forms an orthonormal basis for $L^2(S^{2n_1-1} \times S^{2n_2-1})$. Let $Y_{p_1 q_1 p_2 q_2}^j$ be the restriction of $\tilde{P}_{p_1 q_1 p_2 q_2}^j \in H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2$ i.e

$$\tilde{P}_{p_1 q_1 p_2 q_2}^j(z) = \tilde{P}_{p_1 q_1 p_2 q_2}^j(z^1, z^2) = r_1^{p_1+q_1} r_2^{p_2+q_2} Y_{p_1 q_1 p_2 q_2}^j(\omega^1, \omega^2),$$

where $z^i = r_i \omega^i$, $\omega^i \in S^{2n_i-1}$. Define

$$P_{p_1 q_1 p_2 q_2}^j(z) = \sqrt{\prod_{i=1}^2 \Gamma(n_i) (2|\lambda|)^{-(p_i+q_i)} \frac{\Gamma(m_i - p_i + 1)}{\Gamma(m_i + n_i + q_i)}} \tilde{P}_{p_1 q_1 p_2 q_2}^j(z). \quad (10.5)$$

Then

$$\{P_{p_1 q_1 p_2 q_2}^j(z) : j = 1, 2, \dots, d(p_1, q_1, p_2, q_2); p_i \leq m_i, q_i \in \mathbb{Z}^+; i = 1, 2\}$$

forms an orthonormal basis for $\mathcal{O}^\lambda(V_{m_1 m_2})$.

Proof. Taking $p = q = P_{p_1 q_1 p_2 q_2}^j$ in Lemma 7.6 (b), we get

$$\|\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)\|_{m_1 m_2} = \sqrt{\pi^n (2|\lambda|)^{-n}} \|\theta(P) \psi_{m_1 m_2}\|_2,$$

Therefore by Proposition 10.5, $\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) = 0$, unless $p_1 \leq m_1$, $p_2 \leq m_2$; and if $p_1 \leq m_1$, $p_2 \leq m_2$, writing the right hand side of the above equation in polar coordinates and using the formulae

$$\int_0^\infty [L_k^\alpha(r)]^2 e^{-r} r^\alpha dr = \frac{\Gamma(k + \alpha + 1)}{\Gamma(k + 1)},$$

we can deduce that $\|\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)\|_{m_1 m_2} = 1$. But then, since each $\delta_{p_1 q_1}^1 \otimes \delta_{p_2 q_2}^2$ has a unique (upto a constant multiple) M -fixed vector, the proof follows from Proposition 7.8. \square

Lemma 10.8. *Let $\tilde{P}_{p_1 q_1 p_2 q_2}^j$ and $P_{p_1 q_1 p_2 q_2}^j$ are as in the previous lemma. If f is a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$ satisfying $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for each $\delta \in \hat{K}_M$, then there exist constants $a_{p_1 q_1 p_2 q_2}^j$ such that*

$$f(z) = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} a_{p_1 q_1 p_2 q_2}^j \langle \Pi^\lambda(z), \mathcal{W}^\lambda(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2}^\lambda, \quad (10.6)$$

where the series converges uniformly over compact subsets of \mathbb{C}^n . $a_{p_1 q_1 p_2 q_2}^j$'s satisfy the following :

$$\sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} |a_{p_1 q_1 p_2 q_2}^j|^2 \prod_{i=1}^2 \frac{k_i^{q_i}}{\Gamma(n_i + p_i + q_i)} < \infty, \quad \forall k_1, k_2 \in \mathbb{N}. \quad (10.7)$$

Proof. By Proposition 10.3 (b), we have the expansion, for fixed $r_1, r_2 > 0$,

$$f(z) = f(r_1 \omega^1, r_2 \omega^2) = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} f_{p_1 q_1 p_2 q_2}^j(r_1, r_2) Y_{p_1 q_1 p_2 q_2}^j(\omega^1, \omega^2), \quad (10.8)$$

where the right hand side converges in $L^2(S^{2n_1-1} \times S^{2n_2-1})$. Here

$$f_{p_1 q_1 p_2 q_2}^j(r_1, r_2) = \int_{S^{2n_1-1}} \int_{S^{2n_2-1}} f(r_1 \omega^1, r_2 \omega^2) \overline{Y_{p_1 q_1 p_2 q_2}^j(\omega^1, \omega^2)} d\omega^1 d\omega^2.$$

By a representation theoretic argument it can be shown that (see the proof of Proposition 4.5 in [16]) if $\delta = \delta_{p_1 q_1}^1 \otimes \delta_{p_2 q_2}^2$, $z = (r_1 \omega^1, r_2 \omega^2)$,

$$f_{p_1 q_1 p_2 q_2}^j(r_1, r_2) Y_{p_1 q_1 p_2 q_2}^j(\omega^1, \omega^2) = d(\delta) \int_K f(k \cdot z) (\delta(k^{-1}) Y_{p_1 q_1 p_2 q_2}^j, Y_{p_1 q_1 p_2 q_2}^j) dk.$$

Hence we can conclude that each $f_{p_1 q_1 p_2 q_2}^j(r_1, r_2) Y_{p_1 q_1 p_2 q_2}^j(\omega^1, \omega^2) \in \mathcal{E}^\lambda(\mathbb{C}^n)$ is a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$. But then by Corollary **10.6 (a)**, it follows that (for $z = (z^1, z^2) = (r_1 \omega^1, r_2 \omega^2)$)

$$f_{p_1 q_1 p_2 q_2}^j(r_1, r_2) Y_{p_1 q_1 p_2 q_2}^j(\omega^1, \omega^2) = a_{p_1 q_1 p_2 q_2}^j P_{p_1 q_1 p_2 q_2}^j(z) \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i+q_i} \varphi_{m_i-p_i, \lambda}^{n_i+p_i+q_i-1}(z^i),$$

for some constant $a_{p_1 q_1 p_2 q_2}^j$. Hence (10.6) follows from Corollary **10.6 (b)**. Since $Y_{p_1 q_1 p_2 q_2}^j$ and $P_{p_1 q_1 p_2 q_2}^j$ are related by (10.5), from the above equation we get

$$f_{p_1 q_1 p_2 q_2}^j(r_1, r_2) = a_{p_1 q_1 p_2 q_2}^j b_{p_1 q_1 p_2 q_2}^j \prod_{i=1}^2 r_i^{p_i+q_i} L_{m_i-p_i}^{n_i+p_i+q_i-1}(2|\lambda|r_i^2) e^{-|\lambda|r_i^2},$$

where

$$b_{p_1 q_1 p_2 q_2}^j = \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i+q_i} \sqrt{\Gamma(n_i)(2|\lambda|)^{-(p_i+q_i)} \frac{\Gamma(m_i-p_i+1)}{\Gamma(m_i+n_i+q_i)}}.$$

Now fix $r_1, r_2 > 0$. Since

$$\sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} |f_{p_1 q_1 p_2 q_2}^j(r_1, r_2)|^2 = \|f(r_1 \omega^1, r_2 \omega^2)\|_{L^2(S^{2n_1-1} \times S^{2n_2-1})}^2 < \infty,$$

$$\sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \left[|a_{p_1 q_1 p_2 q_2}^j|^2 \prod_{i=1}^2 \frac{(2|\lambda|r_i^2)^{q_i}}{\Gamma(n_i+p_i+q_i)} \right] \prod_{i=1}^2 \frac{\Gamma(n_i+p_i+q_i)}{\Gamma(m_i+n_i+q_i)} (L_{m_i-p_i}^{n_i+p_i+q_i-1}(2|\lambda|r_i^2))^2 < \infty.$$

Therefore to prove (10.7), it is enough to show that for large q_1, q_2 ,

$$\prod_{i=1}^2 \frac{\Gamma(n_i+p_i+q_i)}{\Gamma(m_i+n_i+q_i)} (L_{m_i-p_i}^{n_i+p_i+q_i-1}(2|\lambda|r_i^2))^2 > c \quad (10.9)$$

for all $p_i \leq m_i$. Now if $\alpha + 1 > 2kt$, then

$$|L_k^\alpha(t)| \geq \frac{1}{2} \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)}.$$

So when $n_i + q_i > 2m_i(2|\lambda|r_i^2)$, then $n_i + p_i + q_i > 2(m_i - p_i)(2|\lambda|r_i^2)$ for all $p_i \leq m_i$ and hence

$$L_{m_i - p_i}^{n_i + p_i + q_i - 1}(2|\lambda|r_i^2) \geq \frac{1}{2} \frac{\Gamma(m_i + q_i + n_i)}{\Gamma(m_i - p_i + 1)\Gamma(n_i + p_i + q_i)} \geq \frac{1}{2} \frac{\Gamma(m_i + q_i + n_i)}{\Gamma(m_i + 1)\Gamma(n_i + p_i + q_i)}.$$

Therefore for all $q_i > 2m_i(2|\lambda|r_i^2) - n_i$ and $p_i \leq m_i$, we have

$$(L_{m_i - p_i}^{n_i + p_i + q_i - 1}(2|\lambda|r_i^2))^2 \geq \frac{1}{4} \frac{\Gamma(m_i + q_i + n_i)}{\Gamma(m_i + 1)^2 \Gamma(n_i + p_i + q_i)},$$

which implies (10.9). Hence the proof is complete. \square

Following [16], for each positive integer k , we define \mathcal{B}_k to be the subspace of operators $S \in \mathcal{O}^\lambda(V_{m_1 m_2})$ for which

$$\sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \|\mathcal{P}_{\mathcal{W}^\lambda(H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2)} S\|_{m_1 m_2}^2 \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}} < \infty,$$

where $\mathcal{P}_{\mathcal{W}^\lambda(H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2)}$ is the projection on $\mathcal{W}^\lambda(H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2)$, that is

$$\mathcal{P}_{\mathcal{W}^\lambda(H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2)} S = \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \langle S, \mathcal{W}^\lambda(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2}^\lambda \mathcal{W}^\lambda(P_{p_1 q_1 p_2 q_2}^j).$$

Then \mathcal{B}_k becomes a Hilbert space if we define the inner product as

$$\langle S_1, S_2 \rangle_{\mathcal{B}_k} = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \langle \mathcal{P}_{\mathcal{W}^\lambda(H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2)} S_1, \mathcal{P}_{\mathcal{W}^\lambda(H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2)} S_2 \rangle_{m_1 m_2}^\lambda \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}}.$$

Note that for each $k \in \mathbb{N}$, $\mathcal{B}_k \subset \mathcal{B}_{k+1}$ and the inclusion $\mathcal{B}_k \hookrightarrow \mathcal{B}_{k+1}$ is continuous.

We define $\mathcal{B} = \cup_{k \in \mathbb{N}} \mathcal{B}_k$ and equip it with the inductive limit topology.

Lemma 10.9. *For each fixed $z \in \mathbb{C}^n$, $\Pi^\lambda(z) \in \mathcal{B}$.*

Proof. Fix $z \in \mathbb{C}^n$.

$$\begin{aligned} \|\mathcal{P}_{\mathcal{W}^\lambda(H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2)} \Pi(z)\|_{m_1 m_2}^2 &= \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \left| \langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2} \right|^2 \\ &= \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \left| P_{p_1 q_1 p_2 q_2}^j(z) \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i + q_i} \varphi_{m_i - p_i, \lambda}^{n_i + p_i + q_i - 1}(z^i) \right|^2. \end{aligned}$$

Since $|Y_{p_1 q_1 p_2 q_2}^j(\omega^1, \omega^2)| \leq c_1 \prod_{i=1}^2 (p_i + q_i)^{n_i-1}$ and

$$|\varphi_{m_i-p_i, \lambda}^{n_i+p_i+q_i-1}(z^i)| \leq c_2 \frac{\Gamma(m_i + n_i + q_i)}{\Gamma(m_i - p_i + 1) \Gamma(n_i + p_i + q_i)},$$

we get

$$\begin{aligned} \|\mathcal{P}_{\mathcal{W}(H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2)} \Pi(z)\|_{m_1 m_2}^2 &\leq c_3 \prod_{i=1}^2 (p_i + q_i)^{2n_i-2} \frac{\Gamma(m_i + n_i + q_i) (2|\lambda| r_i^2)^{p_i+q_i}}{\Gamma(m_i - p_i + 1) (\Gamma(n_i + p_i + q_i))^2} \\ &\leq c_3 \prod_{i=1}^2 (p_i + q_i)^{2n_i} \frac{(m_i + n_i + q_i)^{m_i-p_i} (2|\lambda| r_i^2)^{p_i+q_i}}{\Gamma(m_i - p_i + 1) \Gamma(n_i + p_i + q_i)} \\ &\leq c'_3 \prod_{i=1}^2 \frac{q_i^{2n_i+m_i} (2|\lambda| r_i^2)^{q_i}}{\Gamma(n_i + p_i + q_i)}, \text{ for all } q_i \in \mathbb{Z}^+, p_i \leq m_i. \end{aligned}$$

Now choose k such that $2|\lambda| r_i^2 \leq \frac{k}{2}$. Then

$$\|\mathcal{P}_{\mathcal{W}(H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2)} \Pi(z)\|_{m_1 m_2}^2 \leq c'_3 \prod_{i=1}^2 \frac{q_i^{2n_i+m_i}}{2^{q_i}} \frac{k^{q_i}}{\Gamma(n_i + p_i + q_i)},$$

which implies

$$\|\Pi(z)\|_{\mathcal{B}_k}^2 \leq c'_3 \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \prod_{i=1}^2 \frac{q_i^{2n_i+m_i}}{2^{q_i}} < \infty.$$

Therefore $\Pi(z) \in \mathcal{B}_k$. Hence the proof. \square

Lemma 10.10. *Let \mathcal{B}^* be the dual of \mathcal{B} . If $v \in \mathcal{B}^*$ then*

$$\|v\|_k^2 := \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} |v(\mathcal{W}^\lambda(P_{p_1 q_1 p_2 q_2}^j))|^2 \prod_{i=1}^2 \frac{k^{q_i}}{\Gamma(n_i + p_i + q_i)} < \infty, \quad (10.10)$$

for all $k \in \mathbb{N}$. Conversely if the constants $a_{p_1 q_1 p_2 q_2}^j$'s satisfy

$$\sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} |a_{p_1 q_1 p_2 q_2}^j|^2 \prod_{i=1}^2 \frac{k^{q_i}}{\Gamma(n_i + p_i + q_i)} < \infty \quad (10.11)$$

for all $k \in \mathbb{N}$, then there is a unique $v \in \mathcal{B}^*$ such that $v(\mathcal{W}^\lambda(P_{p_1 q_1 p_2 q_2}^j)) = a_{p_1 q_1 p_2 q_2}^j$.

Proof. Since the topology on \mathcal{B} is the inductive limit topology, $v \in \mathcal{B}^*$ if and only if $v \in \mathcal{B}_k^*$ for all k . Fix a k . Then as \mathcal{B}_k is a Hilbert space, there exists $S_k \in \mathcal{B}_k$ such that $v(S) = \langle S, S_k \rangle_{\mathcal{B}_k}$ for all $S \in \mathcal{B}_k$. Taking $S = \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)$, we get

$$v(\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)) = \langle \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j), S_k \rangle_{\mathcal{B}_k} = \langle \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j), S_k \rangle_{m_1 m_2} \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}}.$$

Since $S_k \in \mathcal{B}_k$,

$$\sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} |\langle S_k, \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2}|^2 \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}} < \infty.$$

Hence (10.10) follows. Conversely, let the constants $a_{p_1 q_1 p_2 q_2}^j$'s satisfy (10.11). Then we can define an operator $S_k \in \mathcal{B}_k$ by

$$\langle \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j), S_k \rangle_{m_1 m_2} = a_{p_1 q_1 p_2 q_2}^j \prod_{i=1}^2 \frac{k^{q_i}}{\Gamma(n_i + p_i + q_i)}.$$

For each $k \in \mathbb{N}$, define $v_k \in \mathcal{B}_k^*$, by $v_k(S) = \langle S, S_k \rangle_{\mathcal{B}_k}$ for all $S \in \mathcal{B}_k$. Note that

$$v_k(S) = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} a_{p_1 q_1 p_2 q_2}^j \langle S, \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2}, \quad S \in \mathcal{B}_k.$$

Therefore for any $S \in \mathcal{B}$, if we define $v(S)$ to be equal to the right hand side of the above equation then $v|_{\mathcal{B}_k} = v_k \in \mathcal{B}_k^*$. Hence $v \in \mathcal{B}^*$. Also note that $v(\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)) = a_{p_1 q_1 p_2 q_2}^j$. Uniqueness of v follows from the fact that

$$\left\{ \sqrt{\prod_{i=1}^2 \frac{k^{q_i}}{\Gamma(n_i + p_i + q_i)}} \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) : j = 1, 2, \dots, d(p_1, q_1, p_2, q_2); p_i \leq m_i, q_i \in \mathbb{Z}^+ \right\}$$

forms an orthonormal basis for \mathcal{B}_k . Hence the proof is complete. \square

Theorem 10.11. *Let f be a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$ such that $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for all $\delta \in \widehat{K}_M$. Then $f(z) = v(\Pi^\lambda(z))$ for a unique $v \in \mathcal{B}^*$. Conversely, if $f(z) = v(\Pi^\lambda(z))$ for some $v \in \mathcal{B}^*$, then f is a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$ and $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for all $\delta \in \widehat{K}_M$.*

Proof. Let $v \in \mathcal{B}^*$ and $f(z) = v(\Pi(z))$. We claim that

$$f(z) = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} v(\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)) \langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2},$$

where the right hand side converges absolutely and uniformly over every compact subset of \mathbb{C}^n . To prove the claim fix $r_i > 0$. Then the proof of Lemma **10.9** shows that there exist $k \in \mathbb{N}$ (depending on r_i) such that $\Pi(z) \in \mathcal{B}_k$ and $\|\Pi(z)\|_{\mathcal{B}_k} < c$ for all $z \in \mathbb{C}^n$ with $|z^i| \leq r_i$. Since $\Pi(z) \in \mathcal{B}_k$, it follows that

$$\sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2} \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)$$

converges to $\Pi(z)$ in the Hilbert space \mathcal{B}_k . Since $v \in \mathcal{B}_k^*$, we get

$$v(\Pi(z)) = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} v(\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)) \langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2}. \quad (10.12)$$

Multiply $v(\mathcal{W}_{m_1 m_2}(P_{p_1 q_1 p_2 q_2}^j))$ by $\Pi_{i=1}^2 k^{q_i} (\Gamma(n_i + p_i + q_i))^{-1}$, $\langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2}$ by $\Pi_{i=1}^2 k^{-q_i} \Gamma(n_i + p_i + q_i)$ and then use the Cauchy-Schwarz inequality to get

$$\begin{aligned} & \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \left| v(\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)) \langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2} \right| \\ &= \leq \|v\|_k \|\Pi(z)\|_{\mathcal{B}_k} \leq c \|v\|_k \end{aligned}$$

for all $z \in \mathbb{C}^n$ such that $|z^i| \leq r_i$. Since $r_i > 0$ was arbitrary, the claim follows. In particular f is a smooth function. Since any $D \in \mathcal{L}_K(\mathbb{C}^n)$ is a polynomial coefficient differential operator we have

$$Df(z) = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} v(\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)) D \left[\langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2} \right]$$

in the distribution sense. But

$$D \left[\langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2} \right] = \mu_{m_1 m_2}(D) \langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2}.$$

Therefore we can conclude that $Df = \mu_{m_1 m_2}(D)f$. Hence f is a joint eigenfunction of all $D \in \mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}$. Now, if $\delta = \delta_{p_1 q_1}^1 \otimes \delta_{p_2 q_2}^2$, equation (10.12) implies that, $\chi_\delta * f = 0$ if $p_1 > m_1$ or $p_2 > m_2$; and when $p_i \leq m_i$ for $i = 1, 2$,

$$\chi_\delta * f = \frac{1}{d(p_1, q_1, p_2, q_2)} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} v(\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)) \langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2}.$$

But, by Proposition 7.15, $\langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2} = \theta(P_{p_1 q_1 p_2 q_2}^j) \psi_{m_1 m_2}$ which clearly equals to $e^{-|\lambda||z|^2}$ times a polynomial. Hence it follows that $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$.

Conversely let f be a joint eigenfunction of all $D \in \mathcal{L}_K(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}$ such that $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for each $\delta \in \widehat{K}_M$. By Lemma 10.8, there exist constants $a_{p_1 q_1 p_2 q_2}^j$ such that

$$f(z) = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} a_{p_1 q_1 p_2 q_2}^j \langle \Pi(z), \mathcal{W}(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2},$$

and $a_{p_1 q_1 p_2 q_2}^j$'s satisfy the following :

$$\sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} |a_{p_1 q_1 p_2 q_2}^j|^2 \prod_{i=1}^2 \frac{k^{q_i}}{\Gamma(n_i + p_i + q_i)} < \infty, \quad \forall k \in \mathbb{N}.$$

Then by the previous lemma there exists $v \in \mathcal{B}^*$ such that $v(\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)) = a_{p_1 q_1 p_2 q_2}^j$, and consequently by (10.12), $f(z) = v(\Pi(z))$.

Now we prove the uniqueness of v which will complete the proof of the theorem. So let $v \in \mathcal{B}^*$ and $v(\Pi(z)) = 0$ for all $z \in \mathbb{C}^n$. We must prove that $v = 0$. It is enough to show that $v(\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)) = 0$ for all $j = 1, 2, \dots, d(p_1, q_1, p_2, q_2); p_i \leq m_i, q_i \in \mathbb{Z}^+$. But this follows, since (10.12) and Corollary 10.6 (b) imply that for each fixed $r_1, r_2 > 0$,

$$\begin{aligned} & \langle v(\Pi(r_1 \cdot, r_2 \cdot)), Y_{p_1 q_1 p_2 q_2}^j \rangle_{L^2(S^{2n_1-1} \times S^{2n_2-1})} \\ &= b_{p_1 q_1 p_2 q_2}^j v(\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)) \prod_{i=1}^2 r_i^{p_i + q_i} L_{m_i - p_i, \lambda}^{n_i + p_i + q_i - 1} (2|\lambda| r_i^2) e^{-|\lambda| r_i^2} \end{aligned}$$

for some non zero constants $b_{p_1 q_1 p_2 q_2}^j$. □

We have already mentioned that the above characterization is analogous to the view point of Thangavelu [16] (see Theorem 4.1 there). Now we make this analogy clear by showing that the above theorem can be reformulated (Theorem **10.12** below), which is similar to Theorem 4.1 in [16]. Consider

$$L_{m_1, m_2}^2(S^{2n_1-1} \times S^{2n_2-1}) := \overline{\text{span}}\{Y_{p_1 q_1 p_2 q_2}^j : j = 1, 2, \dots, d(p_1, q_1, p_2, q_2); p_i \leq m_i, q_i \in \mathbb{Z}^+\}$$

as Hilbert subspace of $L^2(S^{2n_1-1} \times S^{2n_2-1})$. Then the map

$$\mathcal{J} : \mathcal{O}^\lambda(V_{m_1 m_2}) \rightarrow L_{m_1, m_2}^2(S^{2n_1-1} \times S^{2n_2-1})$$

defined by

$$\mathcal{J}(\mathcal{W}^\lambda(P_{p_1 q_1 p_2 q_2}^j)) = Y_{p_1 q_1 p_2 q_2}^j$$

is an Hilbert space isomorphism. Note that $\mathcal{J}(\mathcal{B}_k)$ is the subspace of all functions ϕ in $L_{m_1, m_2}^2(S^{2n_1-1} \times S^{2n_2-1})$ such that

$$\sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \|\phi_{\delta_{p_1 q_1} \otimes \delta_{p_2 q_2}}\|^2 \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}} < \infty,$$

where, for $\omega \in S^{2n_1-1} \times S^{2n_2-1}$,

$$\begin{aligned} \phi_{\delta_{p_1 q_1} \otimes \delta_{p_2 q_2}}(\omega) &:= d(p_1, q_1, p_2, q_2) [\chi_{\delta_{p_1 q_1} \otimes \delta_{p_2 q_2}} * \phi](\omega) \\ &= d(p_1, q_1, p_2, q_2) \int_K \chi_{\delta_{p_1 q_1} \otimes \delta_{p_2 q_2}}(k) \phi(k^{-1} \cdot \omega) dk \\ &= \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \langle \phi, Y_{p_1 q_1 p_2 q_2}^j \rangle Y_{p_1 q_1 p_2 q_2}^j(\omega). \end{aligned}$$

Each $\mathcal{J}(\mathcal{B}_k)$ becomes a Hilbert space with the inner product

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{J}(\mathcal{B}_k)} = \langle \mathcal{J}^{-1} \phi_1, \mathcal{J}^{-1} \phi_2 \rangle_{\mathcal{B}_k}.$$

Explicitly

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{J}(\mathcal{B}_k)} = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \langle (\phi_1)_{\delta_{p_1 q_1} \otimes \delta_{p_2 q_2}}, (\phi_2)_{\delta_{p_1 q_1} \otimes \delta_{p_2 q_2}} \rangle \prod_{i=1}^2 \frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}}.$$

Consider $\mathcal{J}(\mathcal{B}) = \cup_{k \in \mathbb{N}} \mathcal{J}(\mathcal{B}_k)$ and equip this space with the inductive limit topology.

Let

$$\mathcal{P}_{m_1 m_2}^\lambda(z, \omega) = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \langle \Pi^\lambda(z), \mathcal{W}^\lambda(P_{p_1 q_1 p_2 q_2}^j) \rangle_{m_1 m_2}^\lambda Y_{p_1 q_1 p_2 q_2}^j(\omega), \quad (10.13)$$

$\omega \in S^{2n_1-1} \times S^{2n_2-1}$. It is easy to see that $\mathcal{J}(\Pi^\lambda(z)) = \mathcal{P}_{m_1 m_2}^\lambda(z, \cdot)$. Then one can show that Theorem 10.11 is equivalent to the following theorem :

Theorem 10.12. *Let f be a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$ such that $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for all $\delta \in \widehat{K}_M$. Then*

$$f(z) = \int_{S^{2n_1-1} \times S^{2n_2-1}} \mathcal{P}_{m_1 m_2}^\lambda(z, \omega) d\nu(\omega),$$

for a unique $\nu \in \mathcal{B}^*$. Conversely, if

$$f(z) = \int_{S^{2n_1-1} \times S^{2n_2-1}} \mathcal{P}_{m_1 m_2}^\lambda(z, \omega) d\nu(\omega),$$

for some $\nu \in \mathcal{B}^*$, then f is a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$ and $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for all $\delta \in \widehat{K}_M$.

The above theorem gives an integral representation of joint eigenfunctions, where the kernel $\mathcal{P}_{m_1 m_2}^\lambda(z, \omega)$ is given by the series in (10.13). Now we shall give another integral representation, where the kernel can be given explicitly. Fix $a_1, a_2 > 0$ so that

$$L_{m_i - p_i}^{n_i + p_i + q_i - 1}(2|\lambda|a_i^2) \neq 0$$

for all $p_i \leq m_i, q \in \mathbb{Z}^+; i = 1, 2$. Define

$$\begin{aligned} \mathcal{Q}_{m_1 m_2}^\lambda(z, \omega) &= e^{-2i\lambda \operatorname{Im}(z \cdot \overline{(a_1 \omega^1, a_2 \omega^2)})} \psi_{m_1 m_2}^\lambda(z - (a_1 \omega^1, a_2 \omega^2)) \\ &= \pi^{-n} (2|\lambda|)^n \prod_{i=1}^2 e^{2i\lambda a_i \operatorname{Im}(z^i \cdot \bar{\omega}^i)} \varphi_{m_i, \lambda}^{n_i-1}(z^i - a_i \omega^i), \end{aligned}$$

where $z = (z^1, z^2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ and $\omega = (\omega^1, \omega^2) \in S^{2n_1-1} \times S^{2n_2-1}$. For each positive integer k , define \mathcal{A}_k to be the subspace of functions ϕ in $L_{m_1 m_2}^2(S^{2n_1-1} \times S^{2n_2-1})$ for

which

$$\sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \|\phi_{\delta_{p_1 q_2} \otimes \delta_{p_2 q_2}}\|^2 \prod_{i=1}^2 \left[\frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}} \right]^2 < \infty.$$

Each \mathcal{A}_k becomes a Hilbert space with the following inner product :

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{A}_k} = \sum_{\substack{p_1, q_1, p_2, q_2 \in \mathbb{Z}^+ \\ p_1 \leq m_1, p_2 \leq m_2}} \langle (\phi_1)_{\delta_{p_1 q_2} \otimes \delta_{p_2 q_2}}, (\phi_2)_{\delta_{p_1 q_2} \otimes \delta_{p_2 q_2}} \rangle \prod_{i=1}^2 \left[\frac{\Gamma(n_i + p_i + q_i)}{k^{q_i}} \right]^2.$$

We take $\mathcal{A} = \cup_{k \in \mathbb{N}} \mathcal{A}_k$ and equip it with the inductive limit topology. Let \mathcal{A}^* be the dual of \mathcal{A} with respect to this topology. Then we have the following integral representation of joint eigenfunctions of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$.

Theorem 10.13. *Let f be a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$ such that $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for all $\delta \in \widehat{K}_M$. Then*

$$f(z) = \int_{S^{2n_1-1} \times S^{2n_2-1}} \mathcal{Q}_{m_1 m_2}^\lambda(z, \omega) d\nu(\omega),$$

for a unique $\nu \in \mathcal{A}^*$. Conversely, if

$$f(z) = \int_{S^{2n_1-1} \times S^{2n_2-1}} \mathcal{Q}_{m_1 m_2}^\lambda(z, \omega) d\nu(\omega),$$

for some $\nu \in \mathcal{A}^*$, then f is a joint eigenfunction of all $D \in \mathcal{L}_K^\lambda(\mathbb{C}^n)$ with eigenvalue $\mu_{m_1 m_2}^\lambda$ and $\chi_\delta * f \in \mathcal{E}^\lambda(\mathbb{C}^n)$ for all $\delta \in \widehat{K}_M$.

Proof. The theorem can be proved using arguments similar to the proof of Theorem 10.11, once we have the following claim :

$$\int_{S^{2n_1-1} \times S^{2n_2-1}} \mathcal{Q}_{m_1 m_2}^\lambda(z, \omega) Y_{p_1 q_1 p_2 q_2}^j(\omega) d\omega = c_{p_1 q_1 p_2 q_2} (P_{p_1 q_1 p_2 q_2}^j)'(z) \prod_{i=1}^2 \varphi_{m_i - p_i, \lambda}^{n_i + p_i + q_i - 1}(z^i),$$

where $c_{p_1 q_1 p_2 q_2} = 0$ if either $p_1 > m_1$ or $p_2 > m_2$, and for $p_i \leq m_i$, it is given by

$$c_{p_1 q_1 p_2 q_2} = \pi^{-n} (2|\lambda|)^n \prod_{i=1}^2 (2|\lambda|)^{p_i + q_i} \frac{\Gamma(n_i) \Gamma(m_i - p_i + 1)}{\Gamma(m_i + n_i + q_i)} \frac{a_i^{(p_i + q_i)}}{a_i^{2n_i - 1}} L_{m_i - p_i}^{n_i + p_i + q_i - 1}(2|\lambda| a_i^2) e^{-|\lambda| a_i^2}.$$

To prove the claim, first note that we can write

$$\begin{aligned}
& \int_{S^{2n_1-1} \times S^{2n_2-1}} Q_{m_1 m_2}^\lambda(z, \omega) Y_{p_1 q_1 p_2 q_2}^j(\omega) d\omega \\
&= \left[\prod_{i=1}^2 \frac{1}{a_i^{2n_i + p_i + q_i - 1}} \right] (P_{p_1 q_1 p_2 q_2}^j)' d\mu_{a_1, a_2} \times^\lambda \psi_{m_1 m_2}^\lambda(z) \\
&= \left[\sqrt{\prod_{i=1}^2 (2|\lambda|)^{-(p_i + q_i)} \frac{\Gamma(n_i) \Gamma(m_i - p_i + 1)}{\Gamma(m_i + n_i + q_i)} \left(a_i^{2n_i + p_i + q_i - 1} \right)} \right]^{-1} \\
&\times \left[P_{p_1 q_1 p_2 q_2}^j d\mu_{a_1, a_2} \times^\lambda \psi_{m_1 m_2}^\lambda(z) \right],
\end{aligned}$$

where $d\mu_{a_1, a_2}$ is the surface measure on $a_1 S^{2n_1-1} \times a_2 S^{2n_2-1}$. But then the claim follows, if we can prove the following lemma. \square

Lemma 10.14. *Let $a_1, a_2 > 0$ and $d\mu_{a_1, a_2}$ be the surface measure on $a_1 S^{2n_1-1} \times a_2 S^{2n_2-1}$. Let $P \in H_{p_1 q_1}^1 \otimes H_{p_2 q_2}^2$. Then*

$$P d\mu_{a_1, a_2} \times^\lambda \psi_{m_1 m_2}^\lambda = b_{p_1 q_1 p_2 q_2} \pi^{-n} (2|\lambda|)^n P(z) \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i + q_i} \varphi_{m_i - p_i, \lambda}^{n_i + p_i + q_i - 1}(z^i),$$

$$b_{p_1 q_1 p_2 q_2} = \prod_{i=1}^2 (-1)^{q_i} \frac{\Gamma(n_i) \Gamma(m_i - p_i + 1)}{\Gamma(m_i + n_i + q_i)} a_i^{2(p_i + q_i)} L_{m_i - p_i}^{n_i + p_i + q_i - 1} (2|\lambda| a_i^2) e^{-|\lambda| a_i^2},$$

if $p_i \leq m_i$ for all $i = 1, 2$; otherwise $P d\mu_{a_1, a_2} \times^\lambda \psi_{m_1 m_2}^\lambda = 0$.

Proof. Let $\check{\delta} = \delta_{p_1 q_1}^1 \otimes \delta_{p_2 q_2}^2$. Take $P_{p_1 q_1 p_2 q_2}^j$, $j = 1, 2, \dots, d(p_1, q_1, p_2, q_2)$, as Lemma 10.7, so that $\{P_{p_1 q_1 p_2 q_2}^j : j = 1, 2, \dots, d(p_1, q_1, p_2, q_2)\}$ forms a basis for $H_{\check{\delta}}^\vee$. Also we have $\|\mathcal{W}(P_{p_1 q_1 p_2 q_2}^j)\|_{m_1 m_2}^2 = 1$ if $p_i \leq m_i, i = 1, 2$. We can choose suitable bases \mathbf{b} for $V_{\check{\delta}}^\vee$ and \mathbf{e} for $F_{\check{\delta}} = \text{Hom}_K(V_{\check{\delta}}, H_{\check{\delta}})$ so that with respect to these bases $P_{\check{\delta}}^\vee : \mathbb{C}^n \rightarrow \mathcal{M}_{d(\check{\delta}) \times 1}$ can be given as follows : $P_{j1}^{\check{\delta}} = P_{p_1 q_1 p_2 q_2}^j$. Since $\Psi_{m_1 m_2}^{\check{\delta}} = \theta(P_{\check{\delta}}^\vee) \psi_{m_1 m_2}$, by Proposition 10.5, we can say that, $\Psi_{m_1 m_2}^{\check{\delta}} = \tilde{\Psi}_{m_1 m_2}^{\check{\delta}}$ if $p_i \leq m_i$ for all $i = 1, 2$;

otherwise $\Psi_{m_1 m_2}^\delta = 0$. Now let $p_i \leq m_i$ for $i = 1, 2$. Then

$$\begin{aligned} \tilde{A}_{m_1 m_2}^\delta &= A_{m_1 m_2}^\delta = \int_{\mathbb{C}^n} [\Psi_{m_1 m_2}^\delta(z)]^* [\Psi_{m_1 m_2}^\delta(z)] dz \\ &= \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} \left\| \theta(P_{p_1 q_1 p_2 q_2}^j) \psi_{m_1 m_2} \right\|_2^2 \\ &= \pi^{-n} (2|\lambda|)^n d(p_1, q_1, p_2, q_2), \text{ by Lemma 7.6 (b),} \end{aligned}$$

and

$$\tilde{L}_{m_1 m_2}^\delta(z) = \tilde{L}_{m_1 m_2}^\delta(z) = \pi^{-n} (2|\lambda|)^n \prod_{i=1}^2 (-1)^{q_i} (2|\lambda|)^{p_i+q_i} L_{m_i-p_i}^{n_i+p_i+q_i-1} (2|\lambda| |z^i|^2).$$

Also we have

$$\begin{aligned} \Upsilon_\delta^\delta(z) &= [P_\delta^\delta(z)]^* [P_\delta^\delta(z)] \\ &= \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} |P_{p_1 q_1 p_2 q_2}^j(z)|^2 \\ &= \prod_{i=1}^2 (2|\lambda|)^{-(p_i+q_i)} \frac{\Gamma(n_i) \Gamma(m_i - p_i + 1)}{\Gamma(m_i + n_i + q_i)} r_i^{2(p_i+q_i)} \sum_{j=1}^{d(p_1, q_1, p_2, q_2)} |Y_{p_1 q_1 p_2 q_2}^j(\omega)|^2 \\ &= \frac{d(p_1, q_1, p_2, q_2)}{|a_1 S^{2n_1-1} \times a_2 S^{2n_2-1}|} \prod_{i=1}^2 (2|\lambda|)^{-(p_i+q_i)} \frac{\Gamma(n_i) \Gamma(m_i - p_i + 1)}{\Gamma(m_i + n_i + q_i)} r_i^{2(p_i+q_i)}. \end{aligned}$$

Therefore from Theorem 7.10, we can show that, for $p_i \leq m_i, i = 1, 2$,

$$P_\delta^\delta d\mu_{a_1, a_2} \times^\lambda \psi_{m_1 m_2}^\lambda = b_{p_1 q_1 p_2 q_2} \Psi_{m_1 m_2}^\delta,$$

where $b_{p_1 q_1 p_2 q_2}$ is given by

$$b_{p_1 q_1 p_2 q_2} = \prod_{i=1}^2 (-1)^{q_i} \frac{\Gamma(n_i) \Gamma(m_i - p_i + 1)}{\Gamma(m_i + n_i + q_i)} a_i^{2(p_i+q_i)} L_{m_i-p_i}^{n_i+p_i+q_i-1} (2|\lambda| a_i^2) e^{-|\lambda| a_i^2}.$$

Hence the proof follows. \square

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